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Ideals and Solutions of Nonlinear Field Equations

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Families of horizontal ideals of contact manifolds of finite order are studied. Each horizontal ideal is shown to admit an n-dimensional module of Cauchy characteristic vectors that is also a module of annihilators (in the sense of Cartan) of the contact ideal. Since horizontal ideals are generated by 1-forms, any completely integrable horizontal ideal in the family leads to a foliation of the contact manifold by submanifolds of dimension n on which the horizontal ideal vanishes. Explicit conditions are obtained under which an open subset of a leaf of this foliation is the graph of a solution map of the fundamental ideal that characterizes a given system of partial differential equations of finite order with n independent variables. The solution maps are obtained by sequential integration of systems of autonomous ordinary differential equations that are determined by the Cauchy characteristic vector fields for the problem. We show that every smooth solution map can be obtained in this manner. Let $\{V_i | 1 \leq i \leq n\}$ be a basis for the module of Cauchy characteristic vector fields that are in Jacobi normal form. If a subsidiary balance ideal admits each of the n vector fields V_i as a smooth isovector field, then certain leaves of the foliation generated by the corresponding closed horizontal ideal are shown to be graphs of solution maps of the fundamental ideal. A subclass of these constructions agree with those of the Cartan–Kähler theorem. Conditions are also obtained under which every leaf of the foliation is the graph of a solution map. Solving a given system of r partial differential equations with n independent variables on a first-order contact manifold is shown to be equivalent to the problem of constructing a complete system of independent first integrals. Properties of systems of first integrals are analyzed by studying the collection ISO$[A^\alpha_{ij}]$ of all isovectors of the horizontal ideal. We show that ISO$[A^\alpha_{ij}]$ admits the direct sum decomposition $\mathscr{H}^*[A^\alpha_{ij}] \oplus \mathscr{W}[A^\alpha_{ij}]$ as a vector space, where $\mathscr{H}^*[A^\alpha_{ij}]$ is the module of Cauchy characteristics of the horizontal ideal. ISO$[A^\alpha_{ij}]$ also forms a Lie algebra under the standard Lie product, $\mathscr{H}^*[A^\alpha_{ij}]$ and $W[A^\alpha_{ij}]$ are Lie subalgebras of ISO$[A^\alpha_{ij}]$, and $\mathscr{H}^*[A^\alpha_{ij}]$ is an ideal. A change of coordinates that resolves (straightens out) the canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$ is constructed. This change of coordinates is used to reduce the problem of solving the given system of PDE to the problem of root extraction of a system of r functions of n variables, and to establish the existence of solutions to a second-order system of overdetermined PDE that generate the

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subspace $\mathcal{W}[A^{\alpha}_{ij}]$. Similar results are obtained for second-order contact manifolds. Extended canonical transformations are studied. They are shown to provide algorithms for calculating large classes of closed horizontal ideals and a partial analog of classical Hamilton-Jacobi theory.

The advent of nonlinearities in the field equations of non-Abelian gauge theories and related topics has made it much more difficult to obtain exact solutions or even good approximate solutions of these field equations. It has been suggested that the Cartan-Kähler theory should be used in the place of more classical methods. Unfortunately, the Cartan-Kähler theory can lead to computational horrors whose mastery is both difficult and time consuming. This paper presents alternatives to the Cartan-Kähler theory that are more naturally attuned to the structure of the field equations of physical theories (i.e., to equations of balance such as Euler-Lagrange equations). Specific conditions for the existence of local solutions are obtained. Satisfaction of these conditions is shown to provide the necessary information for the explicit construction of solutions by integrating systems of autonomous ordinary differential equations.

1. FUNDAMENTAL IDEALS FOR SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

The n-dimensional manifold of independent variables is denoted by M_n. I assume that M_n is orientable and that a system of local coordinates $\{x^i \mid 1 \le i \le n\}$ has been introduced. In practice, n will usually be 4 or less, and I will often write $\{x, y, z, t\}$ or a subset of these in order to simplify examples. The volume element (basis n-form) of M_n will be denoted by μ. For $n = 4$, the standard orientation gives $\mu = dx \wedge dy \wedge dz \wedge dt$. The conjugate basis for $(n-1)$-forms is given by $\{\mu_i = \partial_i \rfloor \mu \mid 1 \le i \le n\}$ with the properties (Edelen, 1985, Section 3.5)

$$dx^j \wedge \mu_i = \delta^j_i \mu, \qquad d\mu_i = 0$$

Studies of systems of PDE require "place holders" for the dependent variables and their various partial derivatives. These are provided by introducing a *D-order contact manifold* $K_D = M_n \times \mathbb{R}^m$ with local coordinates $\{x^i, q^\alpha, y^\alpha_i, y^\alpha_{ij}, \ldots, y^\alpha_{i_1 i_2 \cdots i_D}\}$ and *contact 1-forms*

$$\begin{gathered} C^\alpha = dq^\alpha - y^\alpha_k\, dx^k, \qquad C^\alpha_i = dy^\alpha_i - y^\alpha_{ik}\, dx^k, \ldots \\ C^\alpha_{i_1 \cdots i_{D-1}} = dy^\alpha_{i_1 \cdots i_{D-1}} - y^\alpha_{i_1 \cdots i_{D-1} k}\, dx^k \end{gathered} \tag{1.1}$$

It may be assumed, without loss of generality, that all y's with more than one lower index are completely symmetric in all lower indices. It will be useful in what follows to define the integer m_D by

$$\dim(K_D) = n + m_D \tag{1.2}$$

I will also assume that there are N dependent variables, so that all Greek indices will range over the integers between 1 and N. Thus, for K_1 we have $m_1 = N + nN$. This allows the introduction of an alternative coordinate designation

$$\{x^i, q^\alpha, y_i^\alpha, \ldots, y_{i_1 \cdots i_D}^\alpha\} = \{x^i, z^A \mid 1 \leq A \leq m_D\}$$

If p_0: $\{x_0^i\}$ is any point in M_n, the m_D-dimensional manifold that is specified by $\{x^i = x_0^i \mid 1 \leq i \leq n\}$ will be referred to as the *fiber* of K_D over p_0 with fiber coordinates $\{z^A\}$.

Let J_n be an open, connected subset of a copy of M_n. If $\Phi: J_n \to K_D$ annihilates each contact 1-form, then the y's become derivatives of the dependent variables with respect to the independent variables on the range of Φ (see Edelen, 1985, Chapter 6). The PDE under study can then be written as a collection of *balance n-forms*

$$B_a = h_a \mu - dW_a^i \wedge \mu_i, \qquad 1 \leq a \leq r \tag{1.3}$$

where $\{h_a, W_a^i \mid 1 \leq a \leq R, 1 \leq i \leq n\}$ are elements of $\Lambda^0(K_D)$. For a first-order contact manifold K_1, we have $h_a = h_a(x^j, q^\alpha, y_j^\alpha)$, $W_a^i = W_a^i(x^j, q^\alpha, y_j^\alpha)$, in which case a map $\Phi \mid x^i = x^i$, $q^\alpha = \phi^\alpha(x^j)$, $y_i^\alpha = \partial_i \phi^\alpha(x^j)$ that annihilates the contact 1-forms will yield

$$\Phi^* B_a = \{h_a(x^j, \phi^\alpha, \partial_j \phi^\alpha) - \frac{d}{dx^i} W_a^i(x^j, \phi^\alpha, \partial_j \phi^\alpha)\}\mu$$

Thus, if the W's all vanish, $\Phi^* B_a = 0$ gives a system of r nonlinear first-order PDE, while for nonvanishing W's, $\Phi^* B_a = 0$ gives a system of r quasilinear second-order PDE. In the case of a contact manifold of order D, $\Phi^* B_a = 0$ will give a system of nonlinear PDE of order D if the W's all vanish, while nonvanishing W's will give a system of quasilinear PDE of order $D+1$. Balance n-forms arise naturally in the calculus of variations of multiple integrals with action n-form $L\mu$, where they are the Euler–Lagrange n-forms generated by the Lagrangian $L \in \Lambda^0(K_1)$; simply take $r = N$, $D = 1$, and

$$h_\alpha = \frac{\partial L}{\partial q^\alpha}, \qquad W_\alpha^i = \frac{\partial L}{\partial y_i^\alpha}$$

They are thus naturally adapted to the study of problems in modern field theory.

This information can be organized in a more efficient manner by using the fact that $\Lambda(K_D)$ is a graded algebra and thus has well-defined ideals. Cartan has shown (Cartan, 1945) that the object of importance is the *fundamental ideal* $\mathcal{I}$ of a system of PDE. This is the closed differential ideal of $\Lambda(K_D)$ that is generated by the contact 1-forms and the balance n-forms;

$$\mathcal{I} = I\{C^\alpha, dC^\alpha, C_i^\alpha, dC_i^\alpha, \ldots, B_a, dB_a\} \tag{1.4}$$

The fundamental ideal contains the *contact ideal*

$$\mathscr{C}_D = I\{C^\alpha, dC^\alpha, \ldots, C^\alpha_{i_1\cdots i_D}, dC^\alpha_{i_1\cdots i_D}\} \tag{1.5}$$

of order D as a closed subideal.

The collection of all *solution maps* of the given system of PDE is given by

$$S = \{\Phi : J_n \to K_D | \Phi^*\mu \neq 0, \Phi^*\mathcal{I} = 0\} \tag{1.6}$$

The requirement $\Phi^*\mu \neq 0$ guarantees that the range of Φ in K_D projects onto M_n as an n-dimensional region; that is, the x's remain independent on the range of Φ. On the other hand, $\Phi^*\mathcal{I} = 0$ if and only if

$$\Phi^* C^\alpha = 0, \qquad \Phi^* C_i^\alpha = 0, \ldots \tag{1.7}$$

and

$$\Phi^* B_a = 0 \tag{1.8}$$

because $\Phi^*\Omega = 0$ implies $\Phi^* d\Omega = 0$. The basic problem is therefore twofold; the solution set S must be shown to be nonvacuous, and methods for the explicit construction of solution maps must be found.

2. CANONICAL SYSTEMS OF VECTOR FIELDS

We noted in the previous section that the contact ideal $\mathscr{C}_D$ of order D is a closed subideal of the fundamental ideal. It turns out that much of the analysis can be based solely on this subideal. This is because the generators of the fundamental ideal that characterize the PDE are represented in this work by the balance n-forms $\{B_a | 1 \leq a \leq r\}$, while the traditional approach represents the PDE under study by 0-forms (Cartan, 1945; Olver, 1986; Pommaret, 1978). Since $\Lambda(K_D)$ is a graded algebra, it proves to be useful to introduce the graded submodules of $\mathscr{C}_D$ over $\Lambda^0(K_D)$ by

$$\mathscr{C}_D^k = \mathscr{C}_D \cap \Lambda^k(K_D) \tag{2.1}$$

An essential aspect of the Cartan–Kähler theory (Cartan, 1945; Kähler, 1949) is the construction of modules of vector fields on K_D that are annihilators of the fundamental ideal. The same proves to be true for the ideal C_D, but the situation is significantly simpler because we will only have to achieve the construction for a "normalized" basis for a module of Cartan annihilators of the contact ideal. The reasons for this will become apparent in what follows.

Let $T(K_D)$ denote the Lie algebra of smooth vector fields on K_D. The following notation will be used for the natural basis elements for $T(K_D)$:

$$\partial_i = \partial/\partial x^i, \qquad \partial_\alpha = \partial/\partial q^\alpha, \qquad \partial^i_\alpha = \partial/\partial y^\alpha_i, \qquad \partial^{ij}_\alpha = \partial/\partial y^\alpha_{ij}, \ldots \tag{2.2}$$

Definition 2.1. A system of n vector fields $\{V_i | 1 \le i \le n\}$ on K_D is said to be a *canonical system* (i.e., a basis for a module of Cartan annihilators of $\mathscr{C}_D$) if and only if the vectors satisfy the normalization conditions

$$V_i \rfloor dx^j = \delta^j_i \tag{2.3}$$

and

$$V_{i_1} \rfloor V_{i_2} \rfloor \cdots \rfloor V_{i_k} \rfloor \mathscr{C}^k_D = 0, \qquad 1 \le k \le n \tag{2.4}$$

Remark. Since there are only n vector fields in a canonical system, the conditions (2.4) will necessarily be satisfied by a canonical system for all $k > n$.

In general, the conditions for a system of n vector fields to be a canonical system depend on the order of the contact manifold. In view of their obvious importance, we will concentrate on contact manifolds of first and second orders. Result for contact manifolds of higher order will follow the same pattern as those given below. The following characterizations of canonical systems are obtained from several theorems established in Section 2 of Edelen (in press).

Theorem 2.1. A system of vector fields $\{V_i | 1 \le i \le n\}$ is a canonical system on a first-order contact manifold if and only if

$$V_i = \partial_i + y^\alpha_i \partial_\alpha + A^\alpha_{ij} \partial^j_\alpha, \qquad 1 \le i \le n \tag{2.5}$$

where the A's are any system of elements of $\Lambda^0(K_1)$ that satisfy the symmetry relations

$$A^\alpha_{ij} = A^\alpha_{ji} \tag{2.6}$$

and we have

$$[V_i, V_j] = \{V_i\langle A^\alpha_{jk}\rangle - V_j\langle A^\alpha_{ik}\rangle\}\partial^k_\alpha \tag{2.7}$$

A first-order contact manifold thus admits an $N(n+1)/2$-fold infinity of canonical systems, and hence $\mathscr{C}_1$ admits an $N(n+1)/2$-fold infinity of modules of Cartan annihilators.

Theorem 2.2. A system of vector fields $\{V_i | 1 \le i \le n\}$ is a canonical system on a second-order contact manifold if and only if

$$V_i = \partial_i + y^\alpha_i \partial_\alpha + y^\alpha_{ij} \partial^j_\alpha + B^\alpha_{ijk} \partial^{jk}_\alpha \tag{2.8}$$

where the B's are any system of elements of $\Lambda^0(K_2)$ that satisfy the symmetry relations

$$B^\alpha_{ijk} = B^\alpha_{jik} = B^\alpha_{jki} \tag{2.9}$$

and we have

$$[V_i, V_j] = \{V_i\langle B^\alpha_{jkm}\rangle - V_j\langle B^\alpha_{ikm}\rangle\}\partial^{km}_\alpha \tag{2.10}$$

A second-order contact manifold thus admits an Nn^2-fold infinity of canonical systems, and $\mathscr{C}_2$ admits an Nn^2-fold infinity of modules of Cartan annihilators.

In order to clarify some of the properties of canonical systems, we recall several standard definitions (Edelen, 1985).

Definition 2.2. A vector field U is a *Cauchy characteristic* of an ideal $\mathscr{N}$ of $\Lambda(K_D)$ if and only if

$$U \rfloor \mathscr{N} \subset \mathscr{N} \tag{2.11}$$

Definition 2.3. A vector field U is an *isovector* of an ideal $\mathscr{N}$ of $\Lambda(K_D)$ if and only if

$$\mathscr{L}_U \mathscr{N} \subset \mathscr{N} \tag{2.12}$$

These lead to the following conclusions.

Theorem 2.3. Let $\{V_i\}$ be a canonical system for a contact manifold of order D. If U is any vector field in the linear span of $\{V_i\}$, then U is neither a Cauchy characteristic vector nor an isovector of the contact ideal of order D.

Proof. For $D=1$, I use Theorem 2.1 to obtain $V_i \rfloor dC^\alpha = dy^\alpha_i - A^\alpha_{ij}\, dx^j$, while for $D=2$, Theorem 2.2 gives $V_i \rfloor dC^\alpha_j = dy^\alpha_{ji} - B^\alpha_{ijm}\, dx^m$. Now, any vector U in the linear span of $\{V_i\}$ has the representation $U = N^i V_i$. We therefore have

$$U \rfloor dC^\alpha = N^i\{dy^\alpha_i - A^\alpha_{ij}\, dx^j\} \notin \mathscr{C}_1$$

for $D=1$, and

$$U \rfloor dC^\alpha_j = N^i\{dy^\alpha_{ji} - B^\alpha_{ijm}\, dx^m\} \notin \mathscr{C}_2$$

for $D=2$. Since similar results can be obtained for any finite value of D, I conclude that U is not a Cauchy characteristic vector of $\mathscr{C}_D$. Noting that $\mathscr{L}_U\mathscr{C}_D = U \rfloor d\mathscr{C}_D + d(U \rfloor \mathscr{C}_D) = U \rfloor d\mathscr{C}_D$ for any U in the linear span of $\{V_i\}$, the previous calculations show that $\mathscr{L}_U C^\alpha \notin \mathscr{C}_1$ and $\mathscr{L}_U C^\alpha_j \notin \mathscr{C}_2$. Since similar results can be obtained for any finite value of D, we conclude that U is not an isovector of $\mathscr{C}_D$. ■

Remark. In jet bundle formulations, K_D loosely corresponds to the Dth jet bundle $\mathscr{J}^{(D)}$ when we take the 0th jet bundle to be the manifold

$\mathscr{J}^{(0)} = M_n \times \mathbb{R}^N$ with local coordinates $\{x^i, q^\alpha \mid 1 \le i \le n, 1 \le \alpha \le N\}$. A vector field on $\mathscr{J}^{(0)}$ has the form $\hat{v} = v^i(x^j, q^\beta)\partial_i + v^\alpha(x^j, q^\beta)\partial_\alpha$. Vector fields on $\mathscr{J}^{(D)}$ are obtained by *prolongation* of vector fields on $\mathscr{J}^{(0)}$ in order to ensure compatibility with jet bundle fibration (see refs. 5 and 7 for the details). A vector field U on $\mathscr{J}^{(D)}$ thus has the form $U = \mathbf{pr}^{(D)}(\hat{v})$. An elementary calculation based on Theorems 2.1 and 2.2 shows that no vector in the linear span of a canonical system is the prolongation of a vector field on $\mathscr{J}^{(0)}$ for $D = 1$ or for $D = 2$. A restriction to the jet bundle formulation thus precludes most of the results that will be established in this paper.

3. REFORMULATION IN TERMS OF HORIZONTAL AND VERTICAL IDEALS

The fact that no vector in the linear span of a canonical system $\{V_i\}$ is either a Cauchy characteristic or an isovector suggests that the wrong ideal of $\Lambda(K_D)$ has been used. I will confine consideration, for the most part, to K_1 and note the corresponding results for K_2 through remarks. The pattern for general values of D should then be clear to the reader, who can supply the details if the need should arise.

Definition 3.1. The *vertical* ideal of $\Lambda(K_1)$ is the closed differential ideal that is defined by

$$\mathscr{V} = I\{dx^i \mid 1 \le i \le n\} \tag{3.1}$$

Remark. The vertical ideal is universal with respect to the choice of D, and hence (3.1) can be used for any value of D. The reason for referring to this ideal as "vertical" is that $\mathscr{V}$ is annihilated on the submanifolds of codimension n that are given by $\{x^i = k^i \mid 1 \le i \le n\}$, where the k's denote constants; that is, $\mathscr{V}$ vanishes when restricted to fibers of K_D. The ideal $\mathscr{V}$ contains $\Lambda(M_n)$, and hence $\mathscr{V}$ characterizes those aspects of $\Lambda(K_D)$ that are inherited from M_n.

Definition 3.2. A *horizontal* ideal of $\Lambda(K_1)$ is defined by

$$\mathscr{H}[A^\alpha_{ij}] = I\{C^\alpha, H^\alpha_i \mid 1 \le \alpha \le N, 1 \le i \le n\} \tag{3.2}$$

with

$$H^\alpha_i = dy^\alpha_i - A^\alpha_{ij}\, dx^j \tag{3.3}$$

for each choice of $A^\alpha_{ij} \in \Lambda^0(K_1)$ that satisfies the symmetry conditions

$$A^\alpha_{ij} = A^\alpha_{ji} \tag{3.4}$$

Remark. For $D = 2$, a horizontal ideal is defined by

$$\mathscr{H}[B^\alpha_{ijk}] = I\{C^\alpha, C^\alpha_i, H^\alpha_{ij} \mid 1 \le \alpha \le N, 1 \le i, j \le n\} \tag{3.5}$$

with

$$H_{ij}^{\alpha} = dy_{ij}^{\alpha} - B_{ijk}^{\alpha}\, dx^k \tag{3.6}$$

for each choice of $B_{ijk}^{\alpha} \in \Lambda^0(K_2)$ that satisfies the symmetry conditions

$$B_{ijk}^{\alpha} = B_{jik}^{\alpha} = B_{jki}^{\alpha} \tag{3.7}$$

Definition 3.3. A horizontal ideal $\mathscr{H}[A_{ij}^{\alpha}]$ serves to define an associated *horizontal module* $\mathscr{H}^*[A_{ij}^{\alpha}]$ of $T(K_1)$ by

$$\mathscr{H}^*[A_{ij}^{\alpha}] = \{U \in T(K_1) \mid U \rfloor \mathscr{H}[A_{ij}^{\alpha}] \subset \mathscr{H}[A_{ij}^{\alpha}]\} \tag{3.8}$$

that is, $\mathscr{H}^*[A_{ij}^{\alpha}]$ is the module of Cauchy characteristic vector fields of $\mathscr{H}[A_{ij}^{\alpha}]$.

Remark. For $D = 2$, (3.8) is replaced by

$$\mathscr{H}^*[B_{ijk}^{\alpha}] = \{U \in T(K_2) \mid U \rfloor \mathscr{H}[B_{ijk}^{\alpha}] \subset \mathscr{H}[B_{ijk}^{\alpha}]\} \tag{3.9}$$

and $\mathscr{H}^*[B_{ijk}^{\alpha}]$ is the module of Cauchy characteristic vector fields of $\mathscr{H}[B_{ijk}^{\alpha}]$.

Theorem 3.1. The horizontal module $\mathscr{H}^*[A_{ij}^{\alpha}]$ admits the canonical system

$$V_i = \partial_i + y_i^{\alpha}\partial_{\alpha} + A_{ij}^{\alpha}\partial_{\alpha}^{j}, \qquad 1 \le i \le n \tag{3.10}$$

as a basis. Hence $\mathscr{H}^*[A_{ij}^{\alpha}]$ is a module of Cartan annihilators of $\mathscr{C}_1$ and

$$V_i \rfloor C^{\alpha} = 0, \qquad V_i \rfloor H_j^{\alpha} = 0 \tag{3.11}$$

Proof. Since $\mathscr{H}[A_{ij}^{\alpha}]$ is generated by the 1-forms $\{C^{\alpha}, H_i^{\alpha}\}$, any element Ω of $\mathscr{H}[A_{ij}^{\alpha}]$ is of the form

$$\Omega = C^{\alpha} \wedge P_{\alpha} + H_i^{\alpha} \wedge Q_{\alpha}^{i}$$

with $\{P_{\alpha}, Q_{\alpha}^{i}\}$ of the same degree. We therefore have

$$U \rfloor \Omega \equiv (U \rfloor C^{\alpha})P_{\alpha} + (U \rfloor H_i^{\alpha})Q_i^{\alpha} \bmod \mathscr{H}[A_{ij}^{\alpha}]$$

and hence

$$U = u^i\partial_i + u^{\alpha}\partial_{\alpha} + u_i^{\alpha}\partial_{\alpha}^{i} \in T(K_1)$$

can belong to $\mathscr{H}^*[A_{ij}^{\alpha}]$ if and only if

$$0 = U \rfloor C^{\alpha} = u^{\alpha} - y_i^{\alpha}u^i, \qquad 0 = U \rfloor H_i^{\alpha} = u_i^{\alpha} - A_{im}^{\alpha}u^m \tag{3.12}$$

It thus follows that any $U \in \mathscr{H}^*[A_{ij}^{\alpha}]$ is of the form

$$U = u^i\{\partial_i + y_i^{\alpha}\partial_{\alpha} + A_{ij}^{\alpha}\partial_{\alpha}^{j}\} = u^i V_i \tag{3.13}$$

with $\{V_i\}$ given by (3.10). Since $A_{ij}^{\alpha} = A_{ji}^{\alpha}$, Theorem 2.1 shows that

$$\{V_i \mid 1 \le i \le n\}$$

is a canonical system. Thus, since the vectors in a canonical system are independent, (3.13) shows that $\{V_i \mid 1 \le i \le n\}$ is a basis for $\mathscr{H}^*[A^\alpha_{ij}]$. Since any system of vector fields of the form (3.10) has been shown to be a basis for a module of Cartan annihilators of $\mathscr{C}_1$, $\mathscr{H}^*[A^\alpha_{ij}]$ is a module of Cartan annihilators of of the contact ideal $\mathscr{C}_1$. ■

Remark. An identical argument shows that for $D=2$, $\mathscr{H}^*[B^\alpha_{ijk}]$ admits the canonical system

$$V_i = \partial_i + y^\alpha_i \partial_\alpha + y^\alpha_{ij}\partial^j_\alpha + B^\alpha_{ijk}\partial^{jk}_\alpha, \qquad 1 \le i \le n \tag{3.14}$$

as a basis.

This result is fundamental in what follows. It tells us how to construct a horizontal ideal $\mathscr{H}[A^\alpha_{ij}]$ of $\Lambda(K_1)$, for any given module $\mathscr{H}^*[A^\alpha_{ij}]$ of Cartan annihilators of the contact ideal $\mathscr{C}_1$, such that $\mathscr{H}^*[A^\alpha_{ij}]$ becomes a module of Cauchy characteristics of $\mathscr{H}[A^\alpha_{ij}]$. The extensive body of information associated with Cauchy characteristics is thus made available for the study of PDE along the lines initiated by Cartan.

It is clear from the definitions of the vertical and horizontal ideals of $\Lambda(K_1)$ that $\Lambda^1(K_1)$ admits the direct sum decomposition

$$\Lambda^1(K_1) = \{\mathscr{V} \cap \Lambda^1(K_1)\} \oplus \{\mathscr{H}[A^\alpha_{ij}] \cap \Lambda^1(K_1)\} \tag{3.15}$$

This leads to the following result, which will be instrumental in what follows.

Lemma 3.1. If f is any smooth function on K_1, then

$$df = V_i\langle f\rangle\, dx^i + (\partial_\beta f) C^\beta + (\partial^j_\beta f) H^\beta_j \tag{3.16}$$

and hence

$$df \cap \mathscr{V} = V_i\langle f\rangle\, dx^i, \qquad df \cap \mathscr{H}[A^\alpha_{ij}] = (\partial_\beta f) C^\beta + (\partial^k_\beta f) H^\beta_k \tag{3.17}$$

Proof. For any $f \in \Lambda^0(K_1)$, we have

$$df = (\partial_k f)\, dx^k + (\partial_\beta f)\, dq^\beta + (\partial^j_\beta f)\, dy^\beta_j \tag{3.18}$$

However, $dq^\beta = C^\beta + y^\beta_k\, dx^k$, $dy^\beta_j = H^\beta_j + A^\beta_{jk}\, dx^k$ by (1.1) and (3.3), and hence an elimination of dq^β and dy^β_j in (3.18) gives (3.16). ■

Theorem 3.2. For any horizontal ideal $\mathscr{H}[A^\alpha_{ij}]$ of $\Lambda(K_1)$ we have

$$dC^\alpha \equiv 0 \bmod \mathscr{H}[A^\alpha_{ij}] \tag{3.19}$$

$$dH^\alpha_i \equiv -\tfrac{1}{2}\{V_m\langle A^\alpha_{ki}\rangle - V_k\langle A^\alpha_{mi}\rangle\}\, dx^m \wedge dx^k \bmod \mathscr{H}[A^\alpha_{ij}] \tag{3.20}$$

where $\{V_i \mid 1 \le i \le n\}$ is the canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$. Hence $\mathscr{H}[A^\alpha_{ij}]$ is a closed differential ideal of $\Lambda(K_1)$ if and only if

$$V_k\langle A^\alpha_{mi}\rangle = V_m\langle A^\alpha_{ki}\rangle \tag{3.21}$$

Proof. It follows directly from $C^\alpha = dq^\alpha - y_i^\alpha\, dx^i$ that $dC^\alpha = -dy_i^\alpha \wedge dx^i$. Since $dy_i^\alpha = H_i^\alpha + A_{ij}^\alpha\, dx^j$ by (3.3), we obtain

$$-dC^\alpha = (H_i^\alpha + A_{ij}^\alpha\, dx^j) \wedge dx^i = H_i^\alpha \wedge dx^i \equiv 0 \bmod \mathscr{H}[A_{ij}^\alpha]$$

when we use the symmetry relations $A_{ij}^\alpha = A_{ji}^\alpha$. In like manner, $dH_i^\alpha = -dA_{ik}^\alpha \wedge dx^k$ from (3.3). Use of Lemma 3.1 to evaluate dA_{ik}^α thus gives

$$\begin{aligned} dH_i^\alpha &= -V_m\langle A_{ik}^\alpha\rangle\, dx^m \wedge dx^k - (\partial_\beta A_{ik}^\alpha) C^\beta \wedge dx^k - (\partial_\beta^j A_{ik}^\alpha) H_j^\beta \wedge dx^k \\ &\equiv -V_m\langle A_{ik}^\alpha\rangle\, dx^m \wedge dx^k \bmod \mathscr{H}[A_{ij}^\alpha] \\ &\equiv -\tfrac{1}{2}\{V_m\langle A_{ki}^\alpha\rangle - V_k\langle A_{mi}^\alpha\rangle\}\, dx^m \wedge dx^k \bmod \mathscr{H}[A_{ij}^\alpha] \end{aligned}$$

when we use the symmetry relations $A_{ik}^\alpha = A_{ki}^\alpha$. ∎

Remark. Similar results hold for $D = 2$ with

$$dC^\alpha \equiv 0, \qquad dC_i^\alpha \equiv 0 \bmod \mathscr{H}[B_{ijk}^\alpha] \tag{3.22}$$

$$dH_{ij}^\alpha \equiv -\tfrac{1}{2}\{V_m\langle B_{kij}^\alpha\rangle - V_k\langle B_{mij}^\alpha\rangle\}\, dx^m \wedge dx^k \bmod \mathscr{H}[B_{ijk}^\alpha] \tag{3.23}$$

Thus, $\mathscr{H}[B_{ijk}^\alpha]$ is a closed differential ideal of $\Lambda(K_2)$ if and only if

$$V_m\langle B_{kij}^\alpha\rangle = V_k\langle B_{mij}^\alpha\rangle \tag{3.24}$$

Theorem 3.3. The horizontal module $\mathscr{H}^*[A_{ij}^\alpha]$ is a module of isovectors of the horizontal ideal $\mathscr{H}[A_{ij}^\alpha]$ if and only if $\mathscr{H}[A_{ij}^\alpha]$ is a closed differential ideal of $\Lambda(K_1)$.

Proof. Since $\mathscr{H}[A_{ij}^\alpha]$ is generated by the 1-forms $\{C^\alpha, H_i^\alpha\}$, any vector U in $\mathscr{H}^*[A_{ij}^\alpha]$ is an isovector of $\mathscr{H}[A_{ij}^\alpha]$ if and only if $\mathscr{L}_U C^\alpha$ and $\mathscr{L}_U H_i^\alpha$ are in $\mathscr{H}[A_{ij}^\alpha]$. Now,

$$\mathscr{L}_U C^\alpha = U \rfloor dC^\alpha + d(U \rfloor C^\alpha) = U \rfloor dC^\alpha \tag{3.25}$$

$$\mathscr{L}_U H_i^\alpha = U \rfloor dH_i^\alpha + d(U \rfloor H_i^\alpha) = U \rfloor dH_i^\alpha \tag{3.26}$$

where I have used the fact that any $U \in \mathscr{H}^*[A_{ij}^\alpha]$ is a Cauchy characteristic of $\mathscr{H}[A_{ij}^\alpha]$ in order to obtain the second equalities. Using the evaluations (3.19) and (3.20) and the fact that any $U \in \mathscr{H}^*[A_{ij}^\alpha]$ is a Cauchy characteristic of $\mathscr{H}[A_{ij}^\alpha]$, it follows that

$$\mathscr{L}_U C^\alpha \equiv 0 \bmod \mathscr{H}[A_{ij}^\alpha] \tag{3.27}$$

and

$$\mathscr{L}_U H_i^\alpha \equiv -\tfrac{1}{2}\{V_m\langle A_{ki}^\alpha\rangle - V_k\langle A_{mi}^\alpha\rangle\} U \rfloor (dx^m \wedge dx^k) \bmod \mathscr{H}[{}_{ij}^\alpha] \tag{3.28}$$

Any $U \in \mathscr{H}^*[A_{ij}^\alpha]$ can be written in the form $U = u^i V_i$ because $\{V_i \mid 1 \le i \le n\}$ is a basis for $\mathscr{H}^*[A_{ij}^\alpha]$, and hence $U \rfloor (dx^m \wedge dx^k) = u^m\, dx^k - u^k\, dx^m \in \mathscr{V}$. Thus, (3.28) shows that $\mathscr{L}_U H_i^\alpha$ is in $\mathscr{H}[A_{ij}^\alpha]$ if and only if

$$V_m\langle A_{ki}^\alpha\rangle = V_k\langle A_{mi}^\alpha\rangle \tag{3.29}$$

Theorem 3.2 shows that these conditions are both necessary and sufficient in order that $\mathscr{H}[A^\alpha_{ij}]$ be a closed differential ideal. ■

Remark. A similar result holds for $D=2$; the horizontal module $\mathscr{H}^*[B^\alpha_{ijk}]$ is a module of isovectors of $\mathscr{H}[B^\alpha_{ijk}]$ if and only if $\mathscr{H}[B^\alpha_{ijk}]$ is a closed differential ideal of $\Lambda(K_2)$.

4. CLOSURE CONDITIONS AND THE RESULTING FOLIATION STRUCTURES

Each choice of the functions $A^\alpha_{ij}(x^k, q^\beta, y^\beta_j)$ satisfying the symmetry relations $A^\alpha_{ij}=A^\alpha_{ji}$ leads to a horizontal ideal $\mathscr{H}[A^\alpha_{ij}]$ of $\Lambda(K_1)$ and to an associated horizontal module $\mathscr{H}^*[A^\alpha_{ij}]$ of $T(K)$ that is both a module of Cauchy characteristic vectors of $\mathscr{H}[A^\alpha_{ij}]$ and a module of Cartan annihilators of $\mathscr{C}_1$. Theorem 3.3 shows that $\mathscr{H}[A^\alpha_{ij}]$ is stable under Lie transport by any vector in $\mathscr{H}^*[A^\alpha_{ij}]$ (i.e., $\mathscr{H}^*[A^\alpha_{ij}]$ is a module of isovectors of $\mathscr{H}[A^\alpha_{ij}]$) if and only if $\mathscr{H}[A^\alpha_{ij}]$ is a closed differential ideal of $\Lambda(K_1)$. On the other hand, Theorem 2.2 shows that $\mathscr{H}[A^\alpha_{ij}]$ is a closed differential ideal if and only if

$$V_i\langle A^\alpha_{jk}\rangle = V_j\langle A^\alpha_{ik}\rangle \tag{4.1}$$

and Theorem 2.1 shows that the conditions (4.1) are satisfied if and only if the canonical basis vectors

$$V_i=\partial_i+y^\alpha_i\partial_\alpha+A^\alpha_{ij}\partial^j_\alpha, \qquad 1\le i\le n \tag{4.2}$$

of $\mathscr{H}^*[A^\alpha_{ij}]$ satisfy

$$[V_i, V_j]=0 \tag{4.3}$$

It thus follows that if $U=u^iV_i$ and $W=w^iV_i$ are any two elements of $\mathscr{H}^*[A^\alpha_{ij}]$ and if $\mathscr{H}[A^\alpha_{ij}]$ is closed, then

$$[U, W]=\{u^iV_i\langle w^j\rangle - w^iV_i\langle u^j\rangle\}V_j\in\mathscr{H}^*[A^\alpha_{ij}] \tag{4.4}$$

Hence, $\mathscr{H}^*[A^\alpha_{ij}]$ forms a Lie subalgebra of $T(K_1)$; that is, $\mathscr{H}^*[A^\alpha_{ij}]$ is *involutive*. This, however, is just a realization of the Cartan theorem that states that the Cauchy characteristic subspace of a closed differential ideal is a Lie subalgebra. Noting that any horizontal ideal is generated by 1-forms, the Frobenius theorem tells us that any closed horizontal ideal is completely integrable, and hence that K_1 is foliated by submanifolds of dimension n on which the closed horizontal ideal vanishes. This section develops certain direct consequences of the Frobenius theorem that are essential to the analysis. Although these results are probably well known to experts, their specific forms do not seem to be available in the standard literature.

I restrict consideration from now on to horizontal ideals that are completely integrable.

Definition 4.1. The collection of all completely integrable horizontal ideals of $\Lambda(K_1)$ is denoted by $\mathfrak{H}(K_1)$; that is,

$$\mathfrak{H}(K_1) = \{\mathscr{H}[A^\alpha_{ij}] \mid d\mathscr{H}[A^\alpha_{ij}] \subset \mathscr{H}[A^\alpha_{ij}]\} \tag{4.5}$$

Theorem 4.1. A horizontal ideal $\mathscr{H}[A^\alpha_{ij}]$ belongs to $\mathfrak{H}(K_1)$ if and only if the A's satisfy

$$A^\alpha_{ij} = A^\alpha_{ji} \tag{4.6}$$

$$V_i\langle A^\alpha_{jk}\rangle = V_j\langle A^\alpha_{ik}\rangle \tag{4.7}$$

where

$$V_i = \partial_i + y^\alpha_i \partial_\alpha + A^\alpha_{ij}\partial^j_\alpha, \qquad 1 \le i \le n \tag{4.8}$$

is the canonical basis for the associated horizontal module $\mathscr{H}^*[A^\alpha_{ij}]$, and (4.7) is equivalent to

$$[V_i, V_j] = 0 \tag{4.9}$$

The set $\mathfrak{H}(K_1)$ is not vacuous, because

$$A^\alpha_{ij} = \partial_i \partial_j \xi^\alpha(x^k) \tag{4.10}$$

satisfies the conditions (4.6) and (4.7) for every smooth choice of the functions $\{\xi^\alpha(x^k) \mid 1 \le \alpha \le N\}$.

Proof. Theorem 3.1 and the Frobenius theorem show that a horizontal ideal $\mathscr{H}[A^\alpha_{ij}]$ is completely integrable if and only if (4.6) and (4.7) hold. It is then a simple computation to see that the A's given by (4.9) satisfy the conditions (4.6) and (4.7), and hence $\mathfrak{H}(K_1)$ is not vacuous. ■

Remark. For $D = 2$, the conditions of this theorem are replaced by

$$\begin{gathered} B^\alpha_{ijk} = B^\alpha_{jik} = B^\alpha_{jki} \\ V_i\langle B^\alpha_{jkm}\rangle = V_j\langle B^\alpha_{ikm}\rangle \end{gathered} \tag{4.11}$$

Theorem 4.2. For each $\mathscr{H}[A^\alpha_{ij}]$ in $\mathfrak{H}(K_1)$, the space K_1 is foliated by manifolds of dimension n that are transverse to the fibers of K_1 and $\mathscr{H}[A^\alpha_{ij}]$ vanishes when restricted to any leaf of this foliation. If $\{V_i \mid 1 \le i \le n\}$ is the canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$, then the leaves of the foliation are given in implicit form by

$$g_\Sigma(x^j, q^\beta, y^\beta_j) = k_\Sigma, \qquad 1 \le \Sigma \le m_1 \tag{4.12}$$

where the functions $\{g_\Sigma \mid 1 \le \Sigma \le m_1\}$ constitute a complete, independent system of primitive integrals of the linear system of partial differential equations

$$V_i\langle g\rangle = 0, \qquad 1 \le i \le n \tag{4.13}$$

Proof. If $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$, then $\mathscr{H}[A^\alpha_{ij}]$ is a closed differential ideal that is generated by m_1 independent 1-forms $\{C^\alpha, H^\alpha_i\}$. Since $\dim(K_1) = n + m_1$, the Frobenius theorem implies that K_1 is foliated by n-dimensional manifolds such that $\mathscr{H}[A^\alpha_{ij}]$ vanishes when restricted to any leaf of this foliation. By definition, $\mathscr{H}^*[A^\alpha_{ij}]$ is the module of Cauchy characteristics of $\mathscr{H}[A^\alpha_{ij}]$ that has the canonical basis $\{V_i \mid \leq i \leq n\}$. Accordingly, any solution of the system of n simultaneous linear partial differential equations (4.13) will be constant on any leaf of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$. The previous discussion has shown that $[V_i, V_j] = 0$ as a consequence of the Cartan theorem and the obvious fact that the canonical basis $\{V_i\}$ for $\mathscr{H}^*[A^\alpha_{ij}]$ is in Jacobi normal form. The fundamental existence theorem for such systems (Edelen, 1985) asserts the existence of $m_1 = \dim(K_1) - n$ functionally independent primitive integrals $\{g_\Sigma \in \Lambda^0(K_1) \mid 1 \leq \Sigma \leq m_1\}$. Thus, any leaf of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$ will satisfy (4.12) for some choice of the constants $\{k_\Sigma \mid 1 \leq \Sigma \leq m_1\}$. It then follows that each leaf of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$ is transverse to the fibers of K_1 because $V_i \rfloor dx^j = \delta^j_i$ and hence $V_n \rfloor V_{n-1} \rfloor \cdots \rfloor V_1 \rfloor \mu = 1$. ■

Remark. The normalization conditions (2.3) serve to select a basis for $\mathscr{H}^*[A^\alpha_{ij}]$ that is in Jacobi normal form. Viewed in this light, the normalization conditions (2.3) are a convenience rather than a restriction, since I could have equally well used any other basis

$$U_i = N^j_i V_j, \qquad 1 \leq i \leq n, \qquad \det(N^j_i) \neq 0 \tag{4.14}$$

I note in passing that such an alternative basis $\{U_i\}$ is *involutive* because $\{V_i\}$ is involutive, and that any system $\{g_\Sigma \mid 1 \leq \Sigma \leq m_1\}$ of independent primitive integrals of $V_i\langle g\rangle = 0$ is also a system of independent primitive integrals of $U_i\langle g\rangle = 0$. Accordingly, a system of independent primitive integrals of $V_i\langle g\rangle = 0$ is naturally associated with the module $\mathscr{H}^*[A^\alpha_{ij}]$ rather than with a particular basis for that module that is used to calculate a system of independent primitive integrals.

Remark. Results identical to Theorem 4.2 hold for any finite value of D.

Theorem 4.3. Let P_0: $\{x^i_0, z^A_0\}$ be any point in K_1, $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$, and $\{V_i \mid 1 \leq i \leq n\}$ be the canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$. The leaf of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$ that passes through P_0 is the orbit of P_0 under the n-parameter Abelian Lie group germ AG_n of point transformations that is generated by the Abelian Lie algebra $\{V_i \mid 1 \leq i \leq n\}$. The group manifold $\mathscr{AG}_n$ of AG_n is linearly homeomorphic to a copy of M_n with local coordinates $\{u^i \mid 1 \leq i \leq n\}$, a parametric representation of the leaf $\mathscr{L}(P_0)$ of the foliation that passes through P_0 is given by

$$x^i = x^i_0 + u^i, \qquad z^A = Z^A(P_0; u^i) \tag{4.15}$$

where the Z's are obtained by sequential integration of the orbital equations of $\{V_i \mid 1 \le i \le n\}$ starting from the point P_0, and (4.15) defines a map Ψ: $\mathscr{A}\mathscr{G}_n \to K_1$ such that

$$\Psi^* \mathscr{H}[A^\alpha_{ij}] = 0 \tag{4.16}$$

$$\Psi^* dx^i = du^i, \qquad \Psi^* dz^A = \Psi^*(V_i \rfloor dz^A)\, du^i \tag{4.17}$$

Proof. Since $\{V_i\}$ is the canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$ and $\mathscr{H}^*[A^\alpha_{ij}]$ is the module of Cauchy characteristics of $\mathscr{H}[A^\alpha_{ij}]$, the vector fields $\{V_i\}$ are tangent to the leaves of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$. Now, the leaves of this foliation are n-dimensional and $\mathscr{H}^*[A^\alpha_{ij}]$ is an n-dimensional, Abelian Lie algebra. Hence, exponentiation of this Abelian Lie algebra gives us the germs of the n-parameter Abelian Lie group AG_n of point transformations that act on K_1. The group AG_n is a group of automorphisms of any leaf of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$, and a simple dimensional argument shows that the action of AG_n on any leaf of the foliation is simply transitive. Thus, if P_0 is any point in K_1, the leaf of the foliation that contains P_0 is the orbit of P_0 under the action of AG_n. Noting that $[V_i, V_j] = 0$, the flows (1-parameter subgroups) of $\{V_i\}$ commute and hence we can compute the orbit of P_0 under the action of AG_n by sequential integration of the orbital equations starting from P_0 (see the Appendix). In particular, we have

$$\frac{dX^i}{du^j} = \delta^i_j, \qquad X^i(u^j = 0) = x^i_0 \tag{4.18}$$

Integrating the system (4.18) gives $X^i = x^i_0 + u^i$, which shows that the group manifold $\mathscr{A}\mathscr{G}_n$ of AG_n is linearly homeomorphic to M_n. If we denote the solutions of the remaining orbital equations of $\{V_i\}$ by $\{Z^A(P_0; u^j)\}$, then $x^i = x^i_0 + u^i$, $z^A = Z^A(P_0; u^j)$ give the AG_n orbit of P_0 in parametric form. This establishes (4.15). The range of the map Ψ that is defined by (4.15) is contained in a leaf of the foliation that is generated by $\mathscr{H}[A^\alpha_{ij}]$. Thus, since $\mathscr{H}[A^\alpha_{ij}]$ vanishes when restricted to any leaf of this foliation, we have established that $\Psi^* \mathscr{H}[A^\alpha_{ij}] = 0$. ■

Remark. A trivial rewriting of the relations $x^i = x^i_0 + u^i$ gives $u^i = x^i - x^i_0$. The latter relation can be viewed as a mapping from M_n to the group manifold of the Abelian Lie group AG_n. It is then a simple matter to recast the results obtained above in terms of a pure gauge theory with gauge group AG_n. The conditions that select the class $\mathfrak{H}(K_1)$ can then be translated into statements about the vanishing of the gauge curvature and torsion.

Theorem 4.4. If $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$, then the map Ψ that is defined in Theorem 4.3 is a solution map of the contact ideal;

$$\Psi^* \mathscr{C}_1 = 0 \tag{4.19}$$

Thus, each leaf of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$ is the graph of a solution map of the contact ideal, and K_1 is foliated by the graphs of these solution maps.

Proof. It was shown in Theorem 4.3 that the map Ψ satisfies $\Psi^*\mathscr{H}[A^\alpha_{ij}] = 0$. Thus, since $\mathscr{H}[A^\alpha_{ij}]$ is generated by $\{C^\alpha, H^\alpha_i\}$, we have $\Psi^* C^\alpha = 0$. Since $\mathscr{C}_1$ is generated by $\{C^\alpha, dC^\alpha\}$ and Ψ^* commutes with exterior differentiation, it follows that $\Psi^*\mathscr{C}_1 = 0$. ∎

Remark. Identical results can be shown to hold for any finite value of D by exactly the same reasoning.

Remark. The result $\Psi^*\mathscr{H}[A^\alpha_{ij}] = 0$ actually carries a significant amount of additional information. Since $\{V_i\}$ is a canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$, we will necessarily have $\Psi^* x^i = x^i_0 + u^i$. If Ψ assigns the q's by the relations $\Psi^* q^\alpha = \psi^\alpha(u^i)$, then $\Psi^* C^\alpha = \Psi^* H^\alpha_i = 0$ gives

$$\Psi^* y^\alpha_i = \frac{\partial \psi^\alpha}{\partial u^i}, \qquad \Psi^* A^\alpha_{ij} = \frac{\partial^2 \psi^\alpha}{\partial u^i\, \partial u^j}$$

and hence the A's provide second derivative information for graphs of solving maps on K_1. A similar argument shows that the B's provide third derivative information for graphs of solving maps on K_2. A specification of the A's by

$$A^\alpha_{ij} = f^\alpha_{ij}(x^k, q^\beta, y^\beta_k)$$

as will always be the case in what follows, will thus pull back to $\mathscr{A}\mathscr{G}_n$ to give

$$\frac{\partial^2 \psi^\alpha}{\partial u^i\, \partial u^j} = f^\alpha_{ij}\left(x^k_0 + u^k, \psi^\beta, \frac{\partial \psi^\beta}{\partial u^k}\right)$$

Thus, since $[V_i, V_j] = 0$, the integrability conditions for these equations are

$$\Psi^* A^\alpha_{ij} = \Psi^* A^\alpha_{ji}, \qquad \Psi^* V_i\langle A^\alpha_{jk}\rangle = \Psi^* V_j\langle A^\alpha_{ik}\rangle$$

Satisfaction of these integrability conditions is guaranteed by the requirement $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$ in view of Theorem 4.1.

5. A BASIC EXISTENCE THEOREM FOR SYSTEMS OF PDE

Any $\mathscr{H}[A^\alpha_{ij}]$ in $\mathfrak{H}(K_1)$ has been shown to lead to a foliation of K_1 by graphs of solution maps of the contact ideal $\mathscr{C}_1$. The ideal $\mathscr{C}_1$ is also a subideal of the fundamental ideal $\mathscr{I} = I\{C^\alpha, dC^\alpha, B_a, dB_a\}$, where the B's are the n-forms given by (1.3) that characterize the system of PDE under study. These facts prompt the following question: is there a leaf of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$ that contains the graph of a solution map of the fundamental ideal? Any leaf of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$ has an associated map Ψ: $\mathscr{A}\mathscr{G}_n \to K_1$ such that

$$\Psi^* \mathscr{C}_1 = 0 \tag{5.1}$$

and

$$\Psi^* dx^i = \Psi^*(V_j \rfloor dx^i)\, du^j = du^i, \qquad \Psi^*\, dz^A = \Psi^*(V_i \rfloor dz^A)\, du^i \tag{5.2}$$

where the u's are local canonical coordinates on the group space $\mathscr{A}\mathscr{G}_n$ of the Abelian group AG_n. Thus, in order for Ψ to be a solution map of the fundamental ideal, we only have to check whether $\Psi^* B_a = 0$, since Ψ^* and exterior differentiation commute.

Theorem 5.1. If $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$, if Ψ is the map associated with a leaf of the foliation that is generated by $\mathscr{H}[A^\alpha_{ij}]$, and if $\{V_i \mid 1 \le i \le n\}$ is the canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$, then

$$\Psi^* B_a = \Psi^*(F_a)\, du^1 \wedge du^2 \wedge \cdots \wedge du^n \tag{5.3}$$

where the F's are elements of $\Lambda^0(K_1)$ that are given by

$$F_a = h_a - V_i\langle W^i_a \rangle, \qquad 1 \le a \le r \tag{5.4}$$

Proof. If $\Omega \in \Lambda^n(K_1)$, use of (5.2) gives

$$\Psi^*\Omega = \Psi^*(V_n \rfloor V_{n-1} \rfloor \cdots \rfloor V_1 \rfloor \Omega)\, du^1 \wedge du^2 \wedge \cdots \wedge du^n \tag{5.5}$$

An elementary calculation based on $B_a = h_a \mu - dW^i_a \wedge \mu_i$ and

$$V_i = \partial_i + y^\alpha_i \partial_\alpha + A^\alpha_{ij} \partial^j_\alpha$$

shows that

$$V_n \rfloor V_{n-1} \rfloor \cdots \rfloor V_1 \rfloor B_a = F_a \tag{5.6}$$

from which the result follows. ■

Theorem 5.2. Let $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$ and let $\mathscr{F}[A^\alpha_{ij}]$ be the point set in K_1 that is defined by

$$\mathscr{F}[A^\alpha_{ij}] = \{P \in K_1 \mid F_a = 0, 1 \le a \le r\} \tag{5.7}$$

If Ψ is the map associated with a leaf of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$ and the graph of Ψ intersects $\mathscr{F}[A^\alpha_{ij}]$ in a point set that is pulled back to an open subset $\mathscr{D}$ of $\mathscr{A}\mathscr{G}_n$ by Ψ^*, then the restriction of the domain of Ψ to $\mathscr{D}$ defines a solution map of the fundamental ideal.

Proof. Theorem 5.1 shows that $\Psi^* B_a = 0$ only when $\Psi^* F_a = 0$. Noting that the F's are 0-forms on K_1, $\Psi^* F_a = 0$ can be satisfied only on those regions $\mathscr{R}$ of K_1 where the graph of Ψ intersects the point set $\mathscr{F}[A^\alpha_{ij}]$. Let

$\mathscr{D} = \Psi^* \mathscr{R}$; then $\mathscr{D}$ can be the domain of a solution map only if $\mathscr{D}$ is an open subset of $\mathscr{A}\mathscr{G}_n$. If $\Psi_{\mathscr{D}}$ denotes the map that results from Ψ by restriction of the domain of Ψ to $\mathscr{D}$, then $\Psi^*_{\mathscr{D}} B_a = 0$ and the result is established. ∎

Remark. Similar results can be obtained for any finite value of D. Since the form of $B_a = h_a \mu - dW_a^m \wedge \mu_m$ is universal over D, there will be no change in the definitions or properties of the F_a's from one value of D to the next.

The point set $\mathscr{F}[A_{ij}^\alpha]$ is the set of simultaneous zeros of the r functions $F_a = h_a - V_j\langle W_a^j \rangle$, and hence it depends on the choice of $\{A_{ij}^\alpha\}$ because $\{V_i\}$ depend on the choice of $\{A_{ij}^\alpha\}$. This explains the notation $\mathscr{F}[A_{ij}^\alpha]$. This notation is used in order to emphasize the fact that we have to test every possible choice of $\{A_{ij}^\alpha\}$ for which $\mathscr{H}[A_{ij}^\alpha]$ is contained in $\mathfrak{H}(K_1)$ in order to use Theorem 5.2 to obtain all solution maps of the fundamental ideal that are accessible by this method.

In the simplest cases, $\mathscr{F}[A_{ij}^\alpha]$ will be a submanifold of K_1 of codimension r. It is well known, however, that the sets of simultaneous zeros of r smooth functions on a manifold K_1 of dimension $n + m_1$ can have a very complicated structure. The conditions of Theorem 5.2 further compound the problem by requiring us to determine intersections of $\mathscr{F}[A_{ij}^\alpha]$ with the leaves of the foliation generated by $\mathscr{H}[A_{ij}^\alpha]$, and then to test whether any such intersection pulls back to $\mathscr{A}\mathscr{G}_n$ to give an open set. It is thus abundantly clear that these tests can fail and we would be unable to establish the existence of a solution map of the fundamental ideal. Further, if the tests associated with Theorem 5.2 are positive, it could happen that only one leaf of the foliation generated by $\mathscr{H}[A_{ij}^\alpha]$ will intersect $\mathscr{F}[A_{ij}^\alpha]$ in a point set that is the image of an open set $\mathscr{D}$ in $\mathscr{A}\mathscr{G}_n$ under the leaf map Ψ. This is also not unexpected, because systems of relatively simple partial differential equations are known to have solution sets that do not foliate the corresponding contact manifold. The reader will perceive that Theorem 5.2 provides for all of the various pathologies that can arise. An obvious question presents itself at this point. Can we find restrictions on the choices of $\{A_{ij}^\alpha\}$ for which these intersection problems become simpler? In particular, can we find whole leaves of the resulting foliation of K_1 that are graphs of solution maps, and when is every leaf of the foliation the graph of a solution map? Some answers to these questions are presented in the next section. Before proceeding to these matters, we note certain strengthenings of the results given by Theorem 5.2.

Theorem 5.3. Let Ψ be a solution map of the fundamental ideal $\mathscr{I}$ that is obtained from Theorem 5.2 with $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K_1)$, and let $\mathscr{I}$ admit an isogroup ISO of symmetry transformations (Edelen, 1980, 1985); then $\text{ISO} \circ \Psi$ is a group of solution maps of the fundamental ideal.

Proof. By definition, ISO is the group of point transformations of K_1 that is obtained by exponentiation of the Lie algebra of isovectors of the fundamental ideal $\mathscr{I}$. Thus, since Ψ is a solution map of $\mathscr{I}$, ISO $\circ \Psi$ is a Lie group of solution maps of $\mathscr{I}$ because

$$(\mathrm{ISO} \circ \Psi)^* \mathscr{I} = \Psi^* \circ \mathrm{ISO}^* \mathscr{I} = \Psi^* \mathscr{I} = 0 \quad \blacksquare$$

I have explicitly restricted consideration to completely integrable horizontal ideals. It might therefore appear that this restriction could eliminate some or all solutions of the fundamental ideal (i.e., some of the solutions of the given system of PDE could be missed). That this is not the case is shown by the following result.

Theorem 5.4. Any smooth (C^2) solution map of the fundamental ideal can be realized as an open, n-dimensional subset of a leaf of the foliation generated by a completely integrable horizontal ideal.

Proof. Let $\Phi: J_n \subset \mathbb{R}^n \to K_1$ be a smooth (C^2) solution map of the fundamental ideal. Since $\Phi^* \mu \neq 0$, Φ has a local presentation

$$\Phi \,|\, x^i = u^i, \qquad q^\alpha = \phi^\alpha(u^j), \qquad y_i^\alpha = \frac{\partial \phi^\alpha(u^k)}{\partial u^i} \tag{5.8}$$

Let

$$A_{ij}^\alpha = \frac{\partial^2 \phi^\alpha(x^k)}{\partial x^i \, \partial x^j} \tag{5.9}$$

Then $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K_1)$, as is easily checked. An integration of the orbital equations of the canonical basis $\{V_i \,|\, 1 \le i \le n\}$ for $\mathscr{H}^*[A_{ij}^\alpha]$ gives the leaf maps

$$\begin{aligned} x^i = x_0^i + u^i, \qquad q^\alpha &= q_0^\alpha + y_{i0}^\alpha u^i + \phi^\alpha(x_0^i + u^i) \\ y_i^\alpha &= y_{i0}^\alpha + \frac{\partial \phi^\alpha(x_0^k + u^k)}{\partial u^i} \end{aligned} \tag{5.10}$$

where the u's are coordinates on a neighborhood J_n of $\mathbb{R}^n$ that contains the origin. It is then easily seen that the solution map with local presentation (5.8) coincides with the leaf map given by (5.10) with all integration constants set equal to zero. $\blacksquare$

Remark. Theorem 5.4 shows that Theorem 5.2, with $\mathscr{H}[A_{ij}^\alpha]$ ranging over all of $\mathfrak{H}(K_1)$, is exhaustive. Thus, if the conditions of Theorem 5.2 are not met for any $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K_1)$, we may conclude that the given system of PDE does not have (smooth) solutions. I note in passing that a result similar to Theorem 5.4 does not seem to have been established in the context of the Cartan–Kähler theory.

6. REDUCTION BY ISOVECTORS OF THE BALANCE IDEAL

Many of the questions associated with the determination of the structure of $\mathscr{F}[A^\alpha_{ij}]$ can be answered by studying yet another ideal of $\Lambda(K_1)$.

Definition 6.1. The *balance ideal* associated with the n-forms $\{B_a \mid 1 \leq a \leq r\}$ over K_1 is given by

$$\mathscr{B}_1[A^\alpha_{ij}] = I\{C^\alpha, H^\alpha_i, B_a\} \tag{6.1}$$

and hence $\mathscr{B}_1[A^\alpha_{ij}]$ admits $\mathscr{H}[A^\alpha_{ij}]$ as a subideal.

Remark. For $D = 2$, the balance ideal is given by

$$\mathscr{B}_2[B^\alpha_{ijk}] = I\{C^\alpha, C^\alpha_i, H^\alpha_{ij}, B_a\} \tag{6.2}$$

Theorem 6.1. If $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$, then the balance ideal $\mathscr{B}_1[A^\alpha_{ij}]$ is a closed differential ideal of $\Lambda(K_1)$ and

$$B_a \equiv F_a\mu \bmod \mathscr{H}[A^\alpha_{ij}] \tag{6.3}$$

so that $F_a\mu$ is the vertical part of B_a.

Proof. If $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$, then $\mathscr{H}[A^\alpha_{ij}]$ is a closed differential ideal. Thus, in view of (6.1), it is sufficient to check that $dB_a \in \mathscr{B}_1[A^\alpha_{ij}]$ for each value of the index a. Noting that (3.1) implies $dB_a = dh_a \wedge \mu$ and each h_a is a 0-form, use of (3.16) gives

$$dB_a = (\partial_\beta h_a) C^\beta \wedge \mu + (\partial^j_\beta h_a) H^\beta_j \wedge \mu \equiv 0 \bmod \mathscr{H}[A^\alpha_{ij}]$$

Use of (3.16) to expand the indicated exterior derivatives in (1.3) thus yields

$$B_a = h_a\mu - V_k\langle W^j_a\rangle\, dx^k \wedge \mu_j \bmod \mathscr{H}[A^\alpha_{ij}]$$

The result then follows from the definitions of the F's given by (5.4), on noting that $dx^k \wedge \mu_j = \delta^k_j \mu$. ■

It is not hard to prove that $\mathscr{H}^*[A^\alpha_{ij}]$ is not a module of Cauchy characteristics of the balance ideal $\mathscr{B}_1[A^\alpha_{ij}]$, so I will not labor the reader with the details. The important question is whether the canonical basis vectors for $\mathscr{H}^*[A^\alpha_{ij}]$ are isovectors of the balance ideal. The following result is therefore useful.

Lemma 6.1. If $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$ and $\{V_i \mid 1 \leq i \leq n\}$ is the canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$, then

$$\mathscr{L}_{V_i} B_a = V_i\langle F_a\rangle\mu \bmod \mathscr{H}[A^\alpha_{ij}] \tag{6.4}$$

Proof. The canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$ has the form

$$V_i = \partial_i + y^\alpha_i \partial_\alpha + A^\alpha_{ij} \partial^j_\alpha$$

and hence

$$\mathscr{L}_{V_i}\mu = 0, \qquad \mathscr{L}_{V_i}\mu_j = 0 \tag{6.5}$$

Thus, Lie differentiation of (1.3) gives

$$\mathscr{L}_{V_i} B_a = V_i\langle h_a\rangle \mu - d(V_i\langle W^j_a\rangle) \wedge \mu_j \tag{6.6}$$

When (3.16) is used to evaluate the indicated exterior derivatives in (6.6) and note is taken that $dx^k \wedge \mu_j = \delta^k_j \mu$, (6.6) is seen to be equivalent to

$$\mathscr{L}_{V_i} B_a \equiv (V_i\langle h_a\rangle - V_j\langle V_i\langle W^j_a\rangle\rangle)\mu \bmod \mathscr{H}[A^\alpha_{ij}] \tag{6.7}$$

Since V_j and V_i commute by Theorem 4.1, I can interchange their order of application in (6.7) to obtain

$$\mathscr{L}_{V_i} B_a \equiv (V_i\langle h_a\rangle - V_i\langle V_j\langle W^j_a\rangle\rangle)\mu \bmod \mathscr{H}[A^\alpha_{ij}] \tag{6.8}$$

The result then follows from the definition of F_a given by (5.4). ■

Theorem 6.2. If $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$ and $\{V_i \mid 1 \le i \le n\}$ is the canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$, then each vector of this canonical basis is an isovector of the balance ideal $\mathscr{B}_1[A^\alpha_{ij}]$ if and only if

$$V_i\langle F_a\rangle = L^b_{ai} F_b, \qquad 1 \le i \le n, \quad 1 \le a \le R \tag{6.9}$$

are satisfied for some choice of the nr^2 elements $\{L^b_{ai}\}$ of $\Lambda^0(K_1)$.

Proof. Since $\mathscr{B}_1[A^\alpha_{ij}] = I\{C^\alpha, H^\alpha_i, B_a\}$ and $\mathscr{H}^*[A^\alpha_{ij}]$ is a module of isovectors of the subideal $\mathscr{H}[A^\alpha_{ij}]$, by Theorem 3.3, it suffices to show that

$$\mathscr{L}_{V_i} B_a \equiv 0 \bmod \mathscr{B}_1[A^\alpha_{ij}]$$

When the evaluations (6.4) are used, the conditions

$$V_i\langle F_a\rangle \mu \equiv L^b_{ai} B_b \bmod \mathscr{H}[A^\alpha_{ij}] \tag{6.10}$$

are obtained. Now, the left-hand sides of (6.10) belong to the vertical ideal $\mathscr{V}$ and Theorem 6.1 shows that $F_a\mu$ is the vertical part of B_a. Accordingly, (6.10) can be satisfied if and only if (6.9) is satisfied. ■

Remark. An elementary calculation shows that every vector field in the module $\mathscr{H}^*[A^\alpha_{ij}]$ is an isovector of the balance ideal when the conditions of Theorem 6.2 are met. Similar results hold for any finite value of D.

Noting that $[V_i, V_j] = 0$, satisfaction of (6.9) implies that the L's satisfy the consistency conditions $V_i\langle L^c_{aj}\rangle + L^b_{aj} L^c_{bi} = V_j\langle L^c_{ai}\rangle + L^b_{ai} L^c_{bj}$. Accordingly, the vector fields

$$\hat{V}_i = V_i + L^b_{ai} F_b \frac{\partial}{\partial F_a}$$

on $K_1 \times \mathbb{R}^r$ satisfy the commutation relations

$$[\hat{V}_i, \hat{V}_j] = 0$$

and hence the quasilinear system (6.9) can be solved in implicit form by converting (6.9) to an equivalent linear system on $K_1 \times \mathbb{R}^r$. Conversely, if the L's are chosen so that they satisfy the consistency conditions given above, then F's can always be found, for a given canonical system $\{V_i \mid 1 \leq i \leq n\}$ that will satisfy (6.9). Many balance n-forms can thus be constructed for which the method will work for any given assignment of a canonical basis. Theorem 6.2 can thus be applied to establish existence of solving maps for many systems of PDE (see Theorem 6.3).

It is essential to realize that the system (6.9) is a system of conditions on the choice of the functions $\{A^\alpha_{ij}\}$. This observation follows from using (5.4) and (2.5) to write out (6.9) and the elements of the canonical basis in their fully expanded forms

$$\begin{gathered} V_i \langle h_a - V_j \langle W^i_a \rangle \rangle = L^b_{ai} \{ h_b + V_j \langle W^j_b \rangle \} \\ V_i = \partial_i + y^\alpha_i \partial_\alpha + A^\alpha_{ij} \partial^j_\alpha \end{gathered} \tag{6.11}$$

and to note that $\{h_a, W^j_a\}$ are given elements of $\Lambda^0(K_1)$ that specify the system of PDE under study.

Theorem 6.3. Let $\mathscr{H}[A^\alpha_{ij}]$ be an element of $\mathfrak{H}(K_1)$ such that the A's satisfy the conditions (6.11), let $\{V_i \mid 1 \leq i \leq n\}$ be the canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$, and let $\mathscr{L}(P_0)$ be the leaf of the foliation of K_1 that contains the point P_0. If P_0 is in $\mathscr{F}[A^\alpha_{ij}]$, then $\mathscr{L}(P_0)$ is contained in $\mathscr{F}[A^\alpha_{ij}]$. If Ψ is the map from $\mathscr{AG}_n$ to K_1 that is constructed by sequential integration of the orbital equations of $\{V_i\}$ starting from P_0, then Ψ is a map from $\mathscr{AG}_n$ to $\mathscr{L}(P_0)$ that is a solution map of the fundamental ideal.

Proof. Since P_0 belongs to $\mathscr{F}[A^\alpha_{ij}]$ by hypothesis, it follows that

$$F_a|_{P_0} = 0, \qquad 1 \leq a \leq r \tag{6.12}$$

Sequential integration of the orbital equations of $\{V_i\}$ starting with the point P_0 (see the Appendix) gives a map Ψ from $\mathscr{AG}_n$ into K_1 such that the image of the origin in $\mathscr{AG}_n$ is the point P_0 and the range of Ψ is the leaf of the foliation that contains P_0. Accordingly, (6.12) gives the evaluations

$$\Psi^*(F_a|_{P_0}) = (\Psi^* F_a)|_{u^i=0} = 0 \tag{6.13}$$

Noting that all V_i restricted to the range of Ψ are tangent to the range of Ψ (i.e., $\{V_i\}$ are tangent to the leaves of the foliation), satisfaction of the conditions (6.11) implies that $\Psi^* F_a$ satisfy

$$\frac{d(\Psi^* F_a)}{du^i} = (\Psi^* L^b_{ai})(\Psi^* F_b) \tag{6.14}$$

where $\{u^i \mid 1 \le i \le n\}$ are local canonical coordinates on the group space $\mathscr{A}\mathscr{G}_n$. Sequential integration of the system (6.14) on $\mathscr{A}\mathscr{G}_n$ subject to the initial data (6.13) gives

$$\Psi^* F_a = 0, \qquad 1 \le a \le r \tag{6.15}$$

Thus, Ψ is a solution map of the fundamental ideal by Theorem 5.2. ■

Remark. Similar results for any finite value of D can be obtained in exactly the same manner.

An examination of the conditions (6.11) shows that there are basically three ways in which they can be satisfied.

Definition 6.1. The subset $\mathfrak{H}_s(K_1)$ of $\mathfrak{H}(K_1)$ that obtains for those choices of $\{A_{ij}^\alpha\}$ for which

$$F_a = h_a - V_j\langle W_a^j\rangle = 0, \qquad 1 \le a \le r \tag{6.16}$$

is termed *special*. Any $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}_s(K_1)$ and the associated $\mathscr{H}^*[A_{ij}^\alpha]$ will also be termed special.

Remark. If the W's all vanish, then $\mathfrak{H}_s(K_1)$ is empty because the resulting equations $h_a = 0$ cannot be satisfied throughout K_1 except in those cases where the given system of PDE is satisfied identically. I assume that such uninteresting systems of PDE have been excluded from the start.

Definition 6.2. The subset $\mathfrak{H}_r(K_1)$ of $\mathfrak{H}(K_1) - \mathfrak{H}_s(K_1)$ that obtains from the choices of $\{A_{ij}^\alpha\}$ for which

$$V_i\langle F_a\rangle = V_i\langle h_a - V_j\langle W_a^j\rangle\rangle = 0, \qquad 1 \le a \le r \tag{6.17}$$

is termed *restricted*. Any $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}_r(K_1)$ and the associated $\mathscr{H}^*[A_{ij}^\alpha]$ will also be termed restricted.

Definition 6.3. The subset $\mathfrak{H}_g(K_1)$ of $\mathfrak{H}(K_1) - \mathfrak{H}_s(K_1) - \mathfrak{H}_r(K_1)$ that obtains from the choices of $\{A_{ij}^\alpha\}$ for which

$$V_i\langle F_a\rangle = L_{ai}^b F_b \tag{6.18}$$

for some not identically zero choice of the functions $\{L_{ai}^b\}$ will be termed *general*. Any $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}_g(K_1)$ and the associated $\mathscr{H}^*[A_{ij}^\alpha]$ will also be termed general.

Theorem 6.4. If $\mathfrak{H}_s(K_1)$ is not vacuous and $\mathscr{H}[A_{ij}^\alpha]$ belongs to $\mathfrak{H}_s(K_1)$, then every leaf of the foliation generated by $\mathscr{H}[A_{ij}^\alpha]$ is the graph of a solution map of the fundamental ideal; that is, K_1 is foliated by solutions of the system of PDE. The conditions that the A's must satisfy in these circumstances are

$$A_{ij}^\alpha = A_{ji}^\alpha, \qquad V_i\langle A_{jk}^\alpha\rangle = V_j\langle A_{ik}^\alpha\rangle \tag{6.19}$$

$$V_i\langle W_a^i\rangle = h_a \tag{6.20}$$

with at least one of the W's not constant on K_1 and

$$V_i = \partial_i + y_i^\alpha \partial_\alpha + A_{ij}^\alpha \partial_\alpha^j$$

Proof. The definition of $\mathfrak{H}_s(K_1)$ shows that the F's vanish throughout K_1 for any $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}_s(K_1)$. Hence, every point in K_1 is a point P_0 for which Theorem 6.3 is applicable. This shows that every leaf of the foliation of K_1 generated by $\mathscr{H}[A_{ij}^\alpha]$ is the graph of a solution map of the fundamental ideal. The conditions (6.19) and (6.20) that the A's must satisfy in order that $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}_s(K_1)$ follow directly from previous results. It is then obvious from (6.20) that at least one of the W's must be nonconstant. ■

Remark. For $D = 2$, the conditions (6.19) are replaced by

$$B_{ijk}^\alpha = B_{jik}^\alpha = B_{jki}^\alpha, \qquad V_i\langle B_{jkm}^\alpha\rangle = V_j\langle B_{ikm}^\alpha\rangle \tag{6.21}$$

with

$$V_i = \partial_i + y_i^\alpha \partial_\alpha + y_{ij}^\alpha \partial_\alpha^j + B_{ijk}^\alpha \partial_\alpha^{jk}$$

Theorem 6.5. If $\mathfrak{H}_r(K_1)$ is not vacuous and $\mathscr{H}[A_{ij}^\alpha]$ belongs to $\mathfrak{H}_r(K_1)$, then the F's are constant in value on any leaf of the foliation of K_1 generated by $\mathscr{H}[A_{ij}^\alpha]$. Thus, any leaf of this foliation on which all of the F's vanish is the graph of a solution map of the fundamental ideal. The conditions that the A's must satisfy in these circumstances are

$$A_{ij}^\alpha = A_{ji}^\alpha, \qquad V_i\langle A_{jk}^\alpha\rangle = V_j\langle A_{ik}^\alpha\rangle \tag{6.22}$$

$$V_i\langle h_a - V_j\langle W_a^j\rangle\rangle = 0 \tag{6.23}$$

with the values of the W's unrestricted and

$$V_i = \partial_i + y_i^\alpha \partial_\alpha + A_{ij}^\alpha \partial_\alpha^j$$

Proof. By definition, $\mathscr{H}[A_{ij}^\alpha]$ belongs to $\mathfrak{H}_r(K_1)$ if and only if the A's are such that $V_i\langle F_a\rangle = 0$, $1 \le a \le r$, $1 \le i \le n$. Thus, each of the F's is a solution of the system of simultaneous linear partial differential equations $\{V_i\langle g\rangle = 0 \mid 1 \le i \le n\}$. The known properties of solutions of such systems show that we must have

$$F_a = f_a(g_\Sigma) \tag{6.24}$$

where $\{g_\Sigma \mid 1 \le \Sigma \le m_1\}$ is a system of independent primitive integrals of the system $V_i\langle g\rangle = 0$. We have shown previously, however, that the leaves of the foliation generated by $\mathscr{H}[A_{ij}^\alpha]$ are given in implicit form by the system of relations

$$g_\Sigma = k_\Sigma, \qquad 1 \le \Sigma \le m_1 \tag{6.25}$$

This shows that the F's are constant in value on the leaves of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$. Theorem 6.3 then shows that any leaf of the foliation on which the F's vanish is the graph of a solution map of the fundamental ideal. In fact, these solution leaves are given in implicit form by the relations

$$f_a(k_\Sigma) = 0 \tag{6.26}$$

and (6.25). The conditions (6.22), (6.23) that the A's must satisfy in order that $\mathscr{H}[A^\alpha_{ij}]$ belong to $\mathfrak{H}_r(K_1)$ follow directly from previously established results. ■

Remark. The Cartan–Kähler theorem for systems of first-order PDE is based on the fundamental ideal $I\{C^\alpha, dC^\alpha, h_a, dh_a\}$. This is equivalent to the closed differential ideal $I\{C^\alpha, dC^\alpha, h_a\mu\}$ in our formulation; that is, we have $W^i_a = 0$. Further, any vector field U that is a Cartan annihilator of the fundamental ideal $I\{C^\alpha, dC^\alpha, h_a, dh_a\}$ must be an annihilator of the contact ideal and satisfy $U \rfloor dh_a = U\langle h_a\rangle = 0$. Since $\{V_i \mid 1 \le i \le n\}$ has been shown to be a basis for a module of Cartan annihilators of the contact ideal, we must accordingly have $V_i\langle h_a\rangle = 0$, $1 \le i \le n$, $1 \le a \le r$ in order for $\{V_i \mid 1 \le i \le n\}$ to be a basis for the module of Cartan annihilators of the fundamental ideal. The Cartan–Kähler theorem on K_1 thus correspond with those problems for which $\mathscr{H}[A^\alpha_{ij}]$ belongs to $\mathfrak{H}_r(K_1)$.

Theorem 6.6. If $\mathfrak{H}_g(K_1)$ is not vacuous and $\mathscr{H}[A^\alpha_{ij}]$ belongs to $\mathfrak{H}_g(K_1)$, then the F's are constant in value only on those leaves of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$ that intersect $\mathscr{F}[A^\alpha_{ij}]$. Thus, any leaf of the foliation that intersects $\mathscr{F}[A^\alpha_{ij}]$ is the graph of a solution map of the fundamental ideal. The conditions that the A's must satisfy under these circumstances are

$$A^\alpha_{ij} = A^\alpha_{ji}, \qquad V_i\langle A^\alpha_{jk}\rangle = V_j\langle A^\alpha_{ik}\rangle \tag{6.27}$$

$$V_i\langle h_a - V_j\langle W^j_a\rangle\rangle = L^b_{ai}(h_b - V_j\langle W^j_b\rangle) \tag{6.28}$$

for some $\{L^b_{ai}\}$ not all zero, and

$$V_i = \partial_i + y^\alpha_i \partial_\alpha + A^\alpha_{ij}\partial^j_\alpha$$

Proof. By definition, $\mathscr{H}[A^\alpha_{ij}]$ will belong to $\mathfrak{H}_g(K_1)$ if and only if

$$V_i\langle F_a\rangle = L^b_{ai} F_b \tag{6.29}$$

for some not identically zero choice of $\{L^b_{ai}\}$. Thus, the F's can be constant in value on a leaf $\mathscr{L}$ of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$ only when all of the F's vanish at some point on $\mathscr{L}$, in which case the F's vanish on the whole leaf. This shows that any $\mathscr{L}$ that intersects $\mathscr{F}[A^\alpha_{ij}]$ is contained in $\mathscr{F}[A^\alpha_{ij}]$, and hence $\mathscr{L}$ is the graph of a solution map of the fundamental ideal by Theorem 6.3. The conditions that the A's must satisfy in order for

$\mathscr{H}[A_{ij}^{\alpha}]$ to belong to $\mathfrak{H}_g(K_1)$ follow directly from previously established results. ∎

Remark. For $D=2$, the conditions that replace (6.27) are given by (6.21)

7. EXAMPLES WITH $N=1$ AND $n=2$

The computations quickly get out of hand, so I will restrict the discussion to cases for which $N=1$ and $n=2$. Since there is only one q, I will drop the Greek indices and use a system of local coordinates $\{x, t, q, y_x, y_t\}$ on K_1 and $\{x, t, q, y_x, y_t, y_{xx}, y_{xt}, y_{tt}\}$ on K_2, where I have used $y_{xt}=y_{tx}$ in the latter. In the interests of simplicity, I will consider problems that have only a single balance form, since $N=1$; that is, the examples will not deal with overdetermined systems. Such systems can be handled in the present context (i.e., the necessary integrability conditions on K_1 are contained in the requirements $A_{ij}^{\alpha}=A_{ji}^{\alpha}$, $V_i\langle A_{jk}^{\alpha}\rangle=V_j\langle A_{ik}^{\alpha}\rangle$), but the calculations can become sufficiently involved as to hide the intrinsic simplicity of the methods. The reader can check that the results for overdetermined systems agree with those obtained by the Cartan method.

For a first-order contact manifold K_1 with $n=2$, $N=1$, a canonical basis is of the form

$$\begin{aligned} V_x &= \partial_x + y_x\partial_q + A\partial^x + B\partial^t \\ V_t &= \partial_t + y_t\partial_q + B\partial^x + C\partial^t \end{aligned} \tag{7.1}$$

with $\partial^x=\partial/\partial y_x$, $\partial^t=\partial/\partial y_t$. The symmetry conditions (4.6) of Theorem 4.1 are satisfied by (7.1), while the conditions (4.7) become

$$V_t\langle A\rangle = V_x\langle B\rangle, \qquad V_t\langle B\rangle = V_x\langle C\rangle \tag{7.2}$$

I will concentrate on special and general structures, since restricted structures are similar to those treated by the Cartan–Kähler theorem.

The Ω-Gordon equation in characteristic coordinates $x=X+cT$, $t=X-cT$ is given by $\partial_x\partial_t\phi=\Omega(\phi)$. This is encoded on K_1 by the 2-form

$$B_1 = \Omega(q)\mu - dy_x \wedge \mu_t \tag{7.3}$$

because $\mu = dx\wedge dt$, $\mu_x = dt$, $\mu_t = -dx$. This is the same as (1.3) with

$$h_1 = \Omega(q), \qquad W_1^x = 0, \qquad W_1^t = y_x \tag{7.4}$$

and hence (5.4) gives

$$F_1 = \Omega(q) - V_t\langle y_x\rangle = \Omega(q) - B \tag{7.5}$$

I can therefore obtain $\mathscr{H}[A, B, C]\in\mathfrak{H}_s(K_1)$ by the choice

$$B = \Omega(q) \tag{7.6}$$

If we set $\dot{\Omega}(q)=d\Omega(q)/dq$, a direct computation shows that (7.2) then reduce to

$$\frac{\partial A}{\partial t}+y_t\frac{\partial A}{\partial q}+\Omega\frac{\partial A}{\partial y_x}+C\frac{\partial A}{\partial y_t}=y_x\dot{\Omega} \tag{7.7}$$

$$\frac{\partial C}{\partial x}+y_x\frac{\partial C}{\partial q}+A\frac{\partial C}{\partial y_x}+\Omega\frac{\partial C}{\partial y_t}=y_t\dot{\Omega} \tag{7.8}$$

For general $\Omega(q)$, a solution of this pair of partial differential equations is given by

$$A=\Omega(q)\frac{y_x}{y_t},\qquad C=\Omega(q)\frac{y_t}{y_x} \tag{7.9}$$

Since $\{V_x, V_t\}$ are then regular except on the submanifold S of K_1 given by $y_x=0$, $y_t=0$, Theorem 6.4 shows that every leaf of the foliation of (K_1-S) that is generated by $\mathscr{H}[A, B, C]$ is the graph of a solution map of the Ω-Gordon equation for any smooth $\Omega(q)$. For the Poincaré form $\Omega(q)=me^{kq}$, (7.7) and (7.8) are also satisfied by

$$A=\frac{k}{2}(y_x)^2+f(x),\qquad C=\frac{k}{2}(y_t)^2+g(t) \tag{7.10}$$

where $\{f, g\}$ are any smooth functions of their indicated arguments. In this case, we have a 2-fold infinity of foliations of K_1 such that each leaf of each foliation is the graph of a solution map of the fundamental ideal.

An interesting problem is obtained from the balance form specification

$$W^x=y_xf(y_x/y_t),\qquad W^t=y_tg(y_x/y_t),\qquad h=0 \tag{7.11}$$

because of the homogeneity in the arguments $\{y_x, y_t\}$. An elementary calculation shows that

$$V_x=\partial_x+y_x\partial_q+y_x^{a+1}\partial^x+y_ty_x^a\partial^t, \tag{7.12}$$

$$V_t=\partial_t+y_t\partial_q+y_ty_x^a\partial^x+y_t^2y_x^{a-1}\partial^t \tag{7.13}$$

form a canonical basis for any choice of the parameter a, and they are such that $[V_x, V_t]=0$. Hence, (5.4) and (7.11) give

$$F=y_x^{a-1}[y_x^2f(y_x/y_t)+y_t^2g(y_x/y_t)] \tag{7.14}$$

and an elementary sequence of computations yields

$$V_x\langle F\rangle=y_x^a(1+a)F,\qquad V_t\langle F\rangle=y_ty_x^{a-1}(1+a)F \tag{7.15}$$

The horizontal ideal $\mathscr{H}[y_x^{a+1}, y_t y_x^a, y_t^2 y_x^{a-1}]$ thus belongs to $\mathfrak{H}_g(K_1)$ for every choice of $\{a, f(y_x/y_t), g(y_x/y_t)\}$. Theorem 6.6 shows that the quasilinear, second-order PDE with balance 2-form

$$B_1 = d(y_x f(y_x/y_t)) \wedge \mu_x + d(y_t g(y_x/y_t)) \wedge \mu_t \tag{7.16}$$

has a solution that passes through any point P of K_1 for which

$$F_P = (y_x^2 f(y_x/y_t) + y_t^2 g(y_x/y_t))_P = 0 \tag{7.17}$$

and that these solutions can be obtained by sequential integration of the orbital equations for $\{V_x, V_t\}$ from the point P. If the functions $\{f, g\}$ stand in the relation $y_x^2 f(y_x/y_t) + y_t^2 g(y_x/y_t) = 0$, then every leaf of the foliation is the graph of a solution map for every value of the parameter a.

A canonical system of vector fields on K_2 has the generic form

$$V_x = \partial_x + y_x \partial_q + y_{xx}\partial^x + y_{xt}\partial^t + A\partial^{xx} + B\partial^{xt} + C\partial^{tt} \tag{7.18}$$

$$V_t = \partial_t + y_t \partial_q + y_{xt}\partial^x + y_{tt}\partial^t + B\partial^{xx} + C\partial^{xt} + D\partial^{tt} \tag{7.19}$$

These forms incorporate the symmetry relations $B^1_{ijk} = B^1_{jik} = B^1_{jki}$ and $y_{ij} = y_{ji}$. The remaining relations in (4.11) for $\mathscr{H}[A, B, C, D] \in \mathfrak{H}(K_2)$ translate into

$$V_t\langle A\rangle = V_x\langle B\rangle, \qquad V_t\langle B\rangle = V_x\langle C\rangle \tag{7.20}$$

$$V_t\langle C\rangle = V_x\langle D\rangle \tag{7.21}$$

Let HOM(k) denote the collection of all functions of $\{y_{xx}, y_{xt}, y_{tt}\}$ that are homogeneous of degree k. Consider those second-order PDE for which

$$B_1 = h(y_{xx}, y_{xt}, y_{tt})\mu \tag{7.22}$$

with $h \in \mathrm{HOM}(k)$. The relations (5.4) thus give

$$F_1 = h(y_{xx}, y_{xt}, y_{tt}) \tag{7.23}$$

and $\mathscr{H}[A, B, C, D] \in \mathfrak{H}_g(K_2)$ if and only if (7.20), (7.21) hold, and

$$V_x\langle h\rangle = L_x h, \qquad V_t\langle h\rangle = L_t h \tag{7.24}$$

for some not identically zero choice of $\{L_x, L_t\}$. If I set

$$A = ay_{xx}, \qquad B = ay_{xt}, \qquad C = ay_{tt}, \qquad da = 0, \qquad a \neq 0 \tag{7.25}$$

then (7.20) are satisfied, while (7.21) is satisfied by any function $D(y_{xx}, y_{xt}, y_{tt}) \in \mathrm{HOM}(1)$. An elementary calculation shows that the first of (7.24) is satisfied with

$$L_x = ak \neq 0 \tag{7.26}$$

while the second of (7.24) is satisfied provided $L_t \in \mathrm{HOM}(0)$ and

$$D = \frac{L_t h - a(y_{xt}\partial^{xx}h + y_{tt}\partial^{xt}h)}{\partial^{tt}h} \tag{7.27}$$

This evaluation of D shows that D is well defined at all points of $K_2 - S$, where S is the submanifold of K_2 where $\partial^{tt}h = 0$, and that $D \in \mathrm{HOM}(1)$ on $K_2 - S$. I have therefore satisfied all of the conditions for $\mathscr{H}[A, B, C, D]$ to belong to $\mathfrak{H}_g(K_2)$. The analog of Theorem 6.6 thus shows that the intersection of any leaf of the foliation generated by $\mathscr{H}[A, B, C, D]$ with $K_2 - S$ that contains a point P_0: $\{x_0, t_0, q_0, y_{x0}, y_{t0}, y_{xx0}, y_{tt0}\}$ such that

$$h(y_{xx0}, y_{xt0}, y_{tt0}) = 0 \tag{7.28}$$

is the graph of a solution map of the fundamental ideal. Thus, for example, we have established the existence of solutions to the Monge-Ampere equation that is characterized by

$$h = y_{xx}y_{tt} - (y_{xt})^2 \tag{7.29}$$

Third-order PDE that are linear in third derivatives can also be analyzed in the K_2 setting. For example, the Korteweg-de Vries equation $\phi_t + \phi\phi_x + \phi_{xxx} = 0$ is encoded on K_2 by

$$B_1 = (y_t + qy_x)\mu + dy_{xx} \wedge \mu_x \tag{7.30}$$

Thus, $h_1 = y_t + qy_x$, $W_1^x = -y_{xx}$, $W_1^t = 0$, and (5.4) gives

$$F_1 = y_t + qy_x + A \tag{7.31}$$

Theorem 6.3 can then be applied provided elements $\{A, B, C, D, L_x, L_t\}$ of $\Lambda^0(K_2)$ can be found such that (7.20), (7.21) are satisfied and

$$V_x\langle F_1\rangle = L_x F_1, \qquad V_t\langle F_1\rangle = L_t F_1 \tag{7.32}$$

Of course, the KdV equation can also be analyzed in K_3 with $h_1 = y_t + qy_x + y_{xxx}$, $W_1^x = W_1^t = 0$. Since the W's all vanish in this description, success can only be achieved for leaves of foliations that are generated by horizontal ideals that belong to $\mathfrak{H}_r(K_3)$ or $\mathfrak{H}_g(K_3)$.

8. PRIMITIVE INTEGRALS AND CONSTRAINTS

If $\mathscr{H}[A_{ij}^\alpha]$ belongs to $\mathfrak{H}(K_1)$, then K_1 is foliated by n-dimensional leaves that are given by

$$g_\Sigma(x^i, q^\alpha, y_i^\alpha) = k_\Sigma, \qquad 1 \le \Sigma \le m_1 \tag{8.1}$$

where $\{g_\Sigma | 1 \le \Sigma \le m_1\}$ is a complete system of independent first integrals of the system of linear PDE $V_i\langle g\rangle = 0$, $1 \le i \le n$. Accordingly, any leaf map,

$$\Psi: \quad J_n \subset \mathbb{R}^n \to K_1 | x^i = u^i, q^\alpha = \Psi^\alpha(u^i), y_i^\alpha = \partial\Psi^\alpha/\partial u^i \tag{8.2}$$

obtained by sequential integration of the orbital equations of $\{V_i\}$, is such that Ψ^* annihilates the horizontal ideal $\mathscr{H}[A_{ij}^\alpha]$. In fact, (8.1) and (8.2) are the manifold-intersection and parametric specifications of the leaves of the foliation, respectively. We therefore have

$$\Psi^* g_\Sigma(x^i, q^\alpha, y_i^\alpha) = k_\Sigma, \qquad 1 \le \Sigma \le m_1 \tag{8.3}$$

for an appropriate choice of the constants $\{k_\Sigma | 1 \le \Sigma \le m_1\}$. Further, since Ψ^* annihilates $\mathcal{H}[A_{ij}^\alpha]$, we have $0 = \Psi^* H_i^\alpha = \Psi^*(dy_i^\alpha - A_{ij}^\alpha)$, and hence

$$\frac{\partial \Psi^\alpha}{\partial u^i \, \partial u^j} = \Psi^* A_{ij}^\alpha(x^k, q^\beta, y_k^\beta) \tag{8.4}$$

which are completely integrable because $[V_i, V_j] = 0$ and $A_{ij}^\alpha = A_{ji}^\alpha$.

The partial differential equations under study are specified by the balance n-forms

$$B_a = h_a \mu - dW_a^i \wedge \mu_i \equiv F_a \mu \bmod \mathcal{H}[A_{ij}^\alpha] \tag{8.5}$$

with

$$F_a = h_a - V_i\langle W_a^i \rangle \tag{8.6}$$

Accordingly, a leaf map Ψ is a solving map for the given system of PDE if

$$\Psi^* F_a = \Psi^*(h_a - V_i\langle W_a^i \rangle) = 0 \tag{8.7}$$

Noting that $\Psi^* g_\Sigma(x^i, q^\alpha, y_i^\alpha) = k_\Sigma$, it follows that

$$\Psi^* g_\Sigma(x^i, q^\alpha, y_i^\alpha) = k_\Sigma, \qquad 1 \le \Sigma \le m_1 \tag{8.8}$$

may be viewed as a *complete system of first integrals* of the given system of PDE for an appropriate choice of the integration constants $\{k_\Sigma\}$; simply note that the independence of the collection $\{g_\Sigma\}$ and the implicit function theorem show that the system (8.8) can be solved for $\{q^\alpha, y_i^\alpha\}$ in terms of the x's and that the y's will be the derivatives of the q's with respect to the x's because the resulting map annihilates the horizontal ideal. Thus, *solving a given system of PDE on K_1 is equivalent to the problem of constructing a complete system of first integrals* (8.8). Partial differential equations and ordinary differential equations are thus seen to be similar, since they may both be viewed as being solved by the construction of a complete system of first integrals.

It was shown in Section 6 that there are three classes of problems in which complete systems of first integrals can be constructed. For the first class, $\mathcal{H}[A_{ij}^\alpha] \in \mathfrak{H}_s(K_1)$, namely special systems, $F_a = 0$, $1 \le a \le r$, on K_1. For this class, every leaf of the foliation is the graph of a solution map, and hence the system of first integrals

$$\Psi^* g_\Sigma(x^i, q^\alpha, y_i^\alpha) = k_\Sigma, \qquad 1 \le \Sigma \le m_1 \tag{8.9}$$

is obtained for every choice of the integration constants $\{k_\Sigma\}$. For the second class, $\mathscr{H}[A^\alpha_{ij}]\in\mathfrak{H}_r(K_1)$, namely restricted systems, $V_i\langle F_a\rangle=0$, $1\le i\le n$. Accordingly, there exist functions $K_a(g_\Sigma)$ such that

$$F_a=K_a(g_\Sigma) \tag{8.10}$$

Under these circumstances, the system of first integrals

$$\Psi^*g_\Sigma(x^i,q^\alpha,y^\alpha_i)=k_\Sigma \tag{8.11}$$

is obtained for those values of the integration constants $\{k_\Sigma\}$ that satisfy

$$K_a(k_\Sigma)=0,\qquad 1\le a\le r \tag{8.12}$$

This is tantamount to the statement that the solutions satisfy the differential constraints

$$\Psi^*F_a=\Psi^*(h_a-V_i\langle W^i_a\rangle)=0,\qquad 1\le a\le r \tag{8.13}$$

If the W's all vanish, then (8.13) simply says that any solution satisfies the original system of PDE $h_a=0$. On the other hand, for nonvanishing W's, the system (8.13) is a system of first-order constraints that the solution of the original system of second-order PDE will necessarily satisfy. For the third class, $\mathscr{H}[A^\alpha_{ij}]\in\mathfrak{H}_g(K_1)$, $V_i\langle F_a\rangle=L^b_{ai}F_b$, and hence

$$F_a=M^b_a(x^j,g_\Sigma)R_b(g_\Sigma),\qquad 1\le a\le r \tag{8.14}$$

where

$$V_i\langle M^c_a\rangle=L^b_{ai}M^c_b,\qquad M^c_b(0^i,g_\Sigma)=\delta^c_b \tag{8.15}$$

Under these circumstances, the system of first integrals

$$\Psi^*g_\Sigma(x^i,q^\alpha,y^\alpha_i)=k_\Sigma \tag{8.16}$$

is obtained for those values of the integration constants $\{k_\Sigma\}$ such that

$$R_a(k_\Sigma)=0,\qquad 1\le a\le r \tag{8.17}$$

9. ISOVECTORS OF THE HORIZONTAL IDEAL

A better understanding of the properties of first integrals can be obtained by studying the system of isovectors of the closed ideal $\mathscr{H}[A^\alpha_{ij}]\in\mathfrak{H}(K_1)$. This is because Theorem 3.3 shows that $\mathscr{H}^*[A^\alpha_{ij}]$ is a module of isovectors of $\mathscr{H}[A^\alpha_{ij}]$, the canonical system

$$V_i=\partial_i+y^\alpha_i\partial_\alpha+A^\alpha_{ij}\partial^j_\alpha,\qquad 1\le i\le n \tag{9.1}$$

is a basis for $\mathscr{H}^*[A^\alpha_{ij}]$, and $\{g_\Sigma\,|\,1\le\Sigma\le m_1\}$ is a complete system of independent first integrals of the system $\{V_i\langle g\rangle=0\,|\,1\le i\le n\}$.

Theorem 9.1. A vector field $U \in T(K_1)$ is an isovector of the horizontal ideal $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$ if and only if it is of the form

$$U = n^i V_i + \eta^\alpha \partial_\alpha + V_i\langle \eta^\alpha \rangle \partial^i_\alpha \tag{9.2}$$

for any choice of the n functions $\{n^i \in \Lambda^0(K_1)\}$ and for any choice of the N functions $\{\eta^\alpha \in \Lambda^0(K_1)\}$ that satisfy

$$V_i V_j\langle \eta^\alpha \rangle = (\eta^\beta \partial_\beta + V_k\langle \eta^\beta \rangle \partial^k_\beta)\langle A^\alpha_{ij} \rangle \tag{9.3}$$

Proof. An arbitrary element U of $T(K_1)$ has the form

$$U = n^i \partial_i + n^\alpha \partial_\alpha + n^\alpha_i \partial^i_\alpha \tag{9.4}$$

with coefficients $\{n^i, n^\alpha, n^\alpha_i \mid 1 \le i \le n, 1 \le \alpha \le N\}$ that are elements of $\Lambda^0(K_1)$. Since $\mathscr{H}[A^\alpha_{ij}] = I\{C^\alpha, H^\alpha_i \mid 1 \le \alpha \le N, 1 \le i \le n\}$ with

$$C^\alpha = dq^\alpha - y^\alpha_k \, dx^k, \qquad H^\alpha_i = dy^\alpha_i - A^\alpha_{ik} \, dx^k \tag{9.5}$$

then U is an isovector of $\mathscr{H}[A^\alpha_{ij}]$ if and only if

$$\mathscr{L}_U C^\alpha \equiv 0, \qquad \mathscr{L}_U H^\alpha_i \equiv 0 \bmod \mathscr{H}[A^\alpha_{ij}] \tag{9.6}$$

With $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$, an elementary calculation shows that

$$dC^\alpha = -H^\alpha_k \wedge dx^k \tag{9.7}$$

and

$$dH^\alpha_i = -(\partial_\beta A^\alpha_{ik}) C^\beta \wedge dx^k - (\partial^j_\beta A^\alpha_{ik}) H^\beta_j \wedge dx^k \tag{9.8}$$

Now, $\mathscr{L}_U C^\alpha = U \rfloor dC^\alpha + d(U \rfloor C^\alpha)$, and hence use of (9.7) gives

$$\mathscr{L}_U C^\alpha = -(U \rfloor H^\alpha_k) \, dx^k + n^k H^\alpha_k + d(U \rfloor C^\alpha) \tag{9.9}$$

Let

$$\eta^\alpha = U \rfloor C^\alpha = n^\alpha - y^\alpha_j n^j \tag{9.10}$$

so that

$$n^\alpha = \eta^\alpha + y^\alpha_j n^j \tag{9.11}$$

Then use of (3.16) to expand the indicated exterior derivative in (9.9) yields

$$\begin{aligned} \mathscr{L}_U C^\alpha &= \{V_k\langle \eta^\alpha \rangle - U \rfloor H^\alpha_k\} \, dx^k + (\partial_\beta \eta^\alpha) C^\beta \\ &\quad + \{n^j \delta^\alpha_\beta + \partial^j_\beta \eta^\alpha\} H^\beta_j \\ &\equiv \{V_k\langle \eta^\alpha \rangle - U \rfloor H^\alpha_k\} \, dx^k \bmod \mathscr{H}[A^\alpha_{ij}] \end{aligned} \tag{9.12}$$

It thus follows that

$$V_k\langle \eta^\alpha \rangle = U \rfloor H^\alpha_k \tag{9.13}$$

in order for U to be an isovector of $\mathscr{H}[A^{\alpha}_{ij}]$. However, (9.2) and (9.5) show that $U \rfloor H^{\alpha}_{k} = n^{\alpha}_{k} - A^{\alpha}_{km} n^{m}$, and hence (9.13) gives

$$n^{\alpha}_{k} = V_{k}\langle \eta^{\alpha} \rangle + A^{\alpha}_{km} n^{m} \tag{9.14}$$

When (9.11) and (9.14) are substituted back into (9.4) and (9.1) and $A^{\alpha}_{ij} = A^{\alpha}_{ji}$ are used, the relations (9.2) are obtained for all choices of the functions $\{n^{i}, \eta^{\alpha} \mid 1 \le i \le n, 1 \le \alpha \le N\}$. A calculation identical to that given above shows that

$$\mathscr{L}_{U} H^{\alpha}_{i} = \{ V_{k}\langle U \rfloor H^{\alpha}_{i} \rangle - (\partial_{\beta} A^{\alpha}_{ik}) \eta^{\beta} - (\partial^{j}_{\beta} A^{\alpha}_{ik}) U \rfloor H^{\beta}_{j} \} \, dx^{k}$$
$$+ \{ \partial_{\beta} \langle U \rfloor H^{\alpha}_{i} \rangle + n^{k} \partial_{\beta} A^{\alpha}_{ik} \} C^{\beta} + \{ \partial^{j}_{\beta} \langle U \rfloor H^{\alpha}_{i} \rangle + n^{k} \partial^{j}_{\beta} A^{\alpha}_{ik} \} H^{\beta}_{j} \tag{9.15}$$

Accordingly, U is an isovector of $\mathscr{H}[A^{\alpha}_{ij}]$ only when

$$V_{k}\langle U \rfloor H^{\alpha}_{i} \rangle = (\partial_{\beta} A^{\alpha}_{ik}) \eta^{\beta} + (\partial^{j}_{\beta} A^{\alpha}_{ik}) U \rfloor H^{\beta}_{j} \tag{9.16}$$

Thus, when (9.13) is used to eliminate $U \rfloor H^{\alpha}_{i}$, *I* obtain (9.3) and the result is established. ■

The system (9.3) is an overdetermined system that will entail integrability conditions. It is not difficult to show that all of the integrability conditions are identically satisfied as a consequence of $A^{\alpha}_{ij} = A^{\alpha}_{ji}$ and $[V_{i}, V_{j}] = 0$. I will not go into this here, since a much simpler method will be presented in Section 14, where a complete parametrization of the collection of all solutions will be obtained.

Corollary 9.1. If U is an isovector of $\mathscr{H}[A^{\alpha}_{ij}] \in \mathfrak{H}(K_{1})$ that is generated by $\{n^{i}, \eta^{\alpha} \mid 1 \le i \le n, 1 \le \alpha \le N\}$, in accordance with the requirements of Theorem 9.1, then

$$\mathscr{L}_{U} C^{\alpha} = \partial_{\beta} \langle \eta^{\alpha} \rangle C^{\beta} + \{ \partial^{j}_{\beta} \langle \eta^{\alpha} \rangle + n^{j} \delta^{\alpha}_{\beta} \} H^{\beta}_{j} \tag{9.17}$$

$$\mathscr{L}_{U} H^{\alpha}_{i} = \{ \partial_{\beta} V_{i} \langle \eta^{\alpha} \rangle + n^{k} \partial_{\beta} A^{\alpha}_{ik} \} C^{\beta}$$
$$+ \{ \partial^{j}_{\beta} V_{i} \langle \eta^{\alpha} \rangle + n^{k} \partial^{j}_{\beta} A^{\alpha}_{ik} \} H^{\beta}_{j} \tag{9.18}$$

Proof. These results follow directly from (9.12), (9.13), and (9.15) upon noting that all terms multiplying dx^{k} in the evaluations of $\mathscr{L}_{U} C^{\alpha}$ and $\mathscr{L}_{U} H^{\alpha}_{i}$ vanish for the required evaluations of n^{α} and n^{α}_{i}. ■

10. VECTOR SPACE PROPERTIES

Let $\mathrm{ISO}[A^{\alpha}_{ij}]$ denote the collection of all isovectors of the horizontal ideal $\mathscr{H}[A^{\alpha}_{ij}] \in \mathfrak{H}(K_{1})$; that is,

$$\mathrm{ISO}[A^{\alpha}_{ij}] = \{ U \in T(K_{1}) \mid \mathscr{L}_{U} \mathscr{H}[A^{\alpha}_{ij}] \subseteq \mathscr{H}[A^{\alpha}_{ij}] \} \tag{10.1}$$

Theorem 9.1 shows that any $U \in \mathrm{ISO}[A^{\alpha}_{ij}]$ is of the form

$$U = n^{i} V_{i} + \eta^{\alpha} \partial_{\alpha} + V_{i} \langle \eta^{\alpha} \rangle \partial^{i}_{\alpha} \tag{10.2}$$

for any $\{n^i \mid 1 \le i \le n\}$ and any $\{\eta^\alpha \mid 1 \le \alpha \le N\}$ that satisfy

$$V_i V_j\langle \eta^\alpha \rangle = (\partial_\beta A^\alpha_{ij})\eta^\beta + (\partial^k_\beta A^\alpha_{ij}) V_k\langle \eta^\beta \rangle \tag{10.3}$$

The question thus naturally arises as to the vector space properties of $\mathrm{ISO}[A^\alpha_{ij}]$.

Clearly, $\eta^\alpha = 0$ satisfies (10.3), in which case we see that $U = n^i V_i$ is in $\mathrm{ISO}[A^\alpha_{ij}]$ for all $\{n^i\} \in \Lambda^0(K_1)$. However, the collection of all such vector fields is the module $\mathscr{H}^*[A^\alpha_{ij}]$ of $T(K_1)$ over $\Lambda^0(K_1)$ that is generated by the canonical basis $\{V_i \mid 1 \le i \le n\}$. The collection of vector fields

$$\mathscr{W}[A^\alpha_{ij}] = \{W_\eta = \eta^\alpha \partial_\alpha + V_i\langle \eta^\alpha \rangle \partial^i_\alpha \ \forall \eta^\alpha \text{ satisfying (10.3)}\} \tag{10.4}$$

must therefore be examined. This means that we need to study the collection of all solutions of the system (10.3). Complete integrability of the system (10.3) and a parametrization of its solution space will be established in Section 14. For our present purposes, what is required is information concerning how various solutions of (10.3) combine. It is clear from inspection that (10.3) is a linear system of second-order PDE for the determination of the functions $\{\eta^\alpha\}$. Accordingly, the collection of all $\{\eta^\alpha\}$ satisfying (10.3) forms a linear space. Hence, $\mathscr{W}[A^\alpha_{ij}]$ is a vector subspace of $T(K_1)$.

Let $\mathscr{P}[A^\alpha_{ij}]$ denote the collection of all simultaneous integrals of the system $V_i\langle g \rangle = 0$, where $\{V_i \mid 1 \le i \le n\}$ is the canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$,

$$\mathscr{P}[A^\alpha_{ij}] = \{f \in \Lambda^0(K_1) \mid V_i\langle f \rangle = 0,\ 1 \le i \le n\} \tag{10.5}$$

Thus, any $f \in \mathscr{P}[A^\alpha_{ij}]$ is of the form

$$f = f(g_\Sigma) \tag{10.6}$$

where $\{g_\Sigma(x^i, q^\alpha, y^\alpha_i) \mid 1 \le \Sigma \le m_1\}$ is a complete system of independent first integrals of the system $V_i\langle g \rangle = 0$. Further, it is clear from the definition of $\mathscr{P}[A^\alpha_{ij}]$ that $\mathscr{P}[A^\alpha_{ij}]$ is an associative algebra under the algebraic operations inherited from $\Lambda^0(K_1)$.

If $f \in \mathscr{P}[A^\alpha_{ij}]$ and $\{\eta^\alpha \mid 1 \le \alpha \le N\}$ is any solution of (10.3), then $\{f\eta^\alpha \mid 1 \le \alpha \le N\}$ is also a solution because $V_i\langle f \rangle = 0$ and hence f will factor to the left in every term in (10.3). This shows that $\mathscr{W}[A^\alpha_{ij}]$ is a module over the associative algebra $\mathscr{P}[A^\alpha_{ij}]$. Now, (10.2) shows that $\mathrm{ISO}[A^\alpha_{ij}]$ is a direct sum of the modules $\mathscr{H}^*[A^\alpha_{ij}]$ and $\mathscr{W}[A^\alpha_{ij}]$, and hence we have the following result.

Theorem 10.1. The collection $\mathrm{ISO}[A^\alpha_{ij}]$ of all isovectors of $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$ is a subspace of $T(K_1)$ that admits the direct sum decomposition

$$\mathrm{ISO}[A^\alpha_{ij}] = \mathscr{H}^*[A^\alpha_{ij}] \oplus \mathscr{W}[A^\alpha_{ij}] \tag{10.7}$$

where $\mathscr{H}^*[A^\alpha_{ij}]$ is the submodule of $T(K_1)$ over $\Lambda^0(K_1)$ that is generated by the canonical basis $\{V_i \mid 1 \le i \le n\}$ and $\mathscr{W}[A^\alpha_{ij}]$ is the linear subspace of

$T(K_1)$ defined by (10.4) that is also a module over the associative algebra $\mathscr{P}[A_{ij}^{\alpha}]$ of functions that are annihilated by the action of all element of $\mathscr{H}^*[A_{ij}^{\alpha}]$.

11. LIE ALGEBRA PROPERTIES OF ISOVECTORS AND THEIR SUBMODULES

For clarity of notation, I will use $[\![\ ,\ ]\!]$ from now on to denote the standard Lie product (commutator) of elements of $T(K_1)$. It is well known that $T(K_1)$ forms a Lie algebra with product $[\![\ ,\ ]\!]$. Since $\mathrm{ISO}[A_{ij}^{\alpha}] \subset T(K_1)$, the Lie product is well defined on pairs of elements of $\mathrm{ISO}[A_{ij}^{\alpha}]$.

Lemma 11.1. The subspace $\mathrm{ISO}[A_{ij}^{\alpha}]$ of $T(K_1)$ forms a Lie algebra with product $[\![\ ,\ ]\!]$, and $\mathrm{ISO}[A_{ij}^{\alpha}]$ is a Lie subalgebra of $T(K_1)$.

Proof. By definition, $\mathrm{ISO}[A_{ij}^{\alpha}]$ is the collection of all elements U of $T(K_1)$ such that $\mathscr{L}_U \mathscr{H}[A_{ij}^{\alpha}] \subset \mathscr{H}[A_{ij}^{\alpha}]$. The known property (Edelen, 1985)

$$\mathscr{L}_{[\![U,V]\!]} = \mathscr{L}_U \mathscr{L}_V - \mathscr{L}_V \mathscr{L}_U \tag{11.1}$$

or Lie derivatives shows that

$$[\![\mathrm{ISO}[A_{ij}^{\alpha}], \mathrm{ISO}[A_{ij}^{\alpha}]]\!] \subset \mathrm{ISO}[A_{ij}^{\alpha}] \tag{11.2}$$

which establishes the result. ■

Theorem 10.1 has shown that $\mathrm{ISO}[A_{ij}^{\alpha}] = \mathscr{H}^*[A_{ij}^{\alpha}] \oplus \mathscr{W}[A_{ij}^{\alpha}]$, and hence the question naturally arises as to how the Lie algebra of $\mathrm{ISO}[A_{ij}^{\alpha}]$ partitions with respect to this direct sum decomposition of modules.

Lemma 11.2. For any $\mathscr{H}[A_{ij}^{\alpha}] \in \mathfrak{H}(K_1)$, the module $\mathscr{H}^*[A_{ij}^{\alpha}]$ forms a Lie subalgebra of $\mathrm{ISO}[A_{ij}^{\alpha}]$,

$$[\![\mathscr{H}^*[A_{ij}^{\alpha}], \mathscr{H}^*[A_{ij}^{\alpha}]]\!] \subset \mathscr{H}^*[A_{ij}^{\alpha}] \tag{11.3}$$

If $U_1 = n_1^i V_i$ and $U_2 = n_2^i V_i$ are any two elements of $\mathscr{H}^*[A_{ij}^{\alpha}]$, then

$$[\![U_1, U_2]\!] = U_3 = n_3^i V_i \tag{11.4}$$

with

$$n_3^i = U_1\langle n_2^i \rangle - U_2\langle n_1^i \rangle \tag{11.5}$$

Proof. Recalling that $\{V_i \mid 1 \le i \le n\}$ is the canonical basis for $\mathscr{H}^*[A_{ij}^{\alpha}]$, Theorem 4.1 shows that for any $\mathscr{H}[A_{ij}^{\alpha}] \in \mathfrak{H}(K_1)$ we have $[\![V_i, V_j]\!] = 0$. An elementary calculation then yields

$$[\![U_1, U_2]\!] = [\![n_1^i V_i, n_2^j V_j]\!] = n_3^i V_i \in \mathscr{H}^*[A_{ij}^{\alpha}]$$

with $\{n_3^i\}$ given by (11.5). ■

Remark. Expanding the indicated derivations in (11.5) leads naturally to the definition of a product $\{\cdot|\cdot\}$ on n-tuples of elements of $\Lambda^0(K_1)$:

$$\{n_1|n_2\}^i = n_1^j V_j\langle n_2^i\rangle - n_2^j V_j\langle n_1^i\rangle \tag{11.6}$$

We can then write (11.5) in the equivalent form

$$n_3^i = \{n_1|n_2\}^i \tag{11.7}$$

It is then an easy matter to see that the collection of all n-tuples of elements of Λ^0 equipped with the product $\{\cdot|\cdot\}$ becomes a Poisson algebra because we obviously have

$$\{n_1|n_2\} = -\{n_2|n_1\} \tag{11.8}$$

while

$$\{n_1|\{n_2|n_3\}\} + \{n_2|\{n_3|n_1\}\} + \{n_3|\{n_1|n_2\}\} = 0 \tag{11.9}$$

follows from

$$\mathscr{L}_{[\![n_1^i V_i, n_2^j V_j]\!]} = \mathscr{L}_{\{n_1,n_2\}^k V_k}$$

and the Jacobi identity.

Lemma 11.3. If

$$U = n^j V_j + \eta^\alpha \partial_\alpha + V_k\langle\eta^\alpha\rangle\partial_\alpha^k \tag{11.10}$$

if $\{\eta^\alpha \mid 1 \le \alpha \le N\}$ is any solution of (10.3), and if $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K_1)$, so that U is an element of $\mathrm{ISO}[A_{ij}^\alpha]$ in general position, then

$$[\![V_i, U]\!] = V_i\langle n^j\rangle V_j \in \mathscr{H}^*[A_{ij}^\alpha] \tag{11.11}$$

Proof. A direct computation based on (11.10) shows that

$$[\![V_i, U]\!] = V_i\langle n^j\rangle V_j + \{V_i V_j\langle\eta^\alpha\rangle - \eta^\alpha\partial_\alpha A_{ij}^\beta$$
$$- V_k\langle\eta^\alpha\rangle\partial_\alpha^k A_{ij}^\beta\}\partial_\beta^j$$

However, the quantities inside the braces all vanish as a consequence of the fact that $\{\eta^\alpha\}$ satisfies (10.3), and hence we obtain (11.11). ■

It is of interest to note in passing that this calculation shows that (11.11) will hold for any U of the form given by (11.10) only when $\{\eta^\alpha\}$ satisfies the system (10.3).

Remark. An immediate consequence of this result is that $\mathscr{L}_W[\![V_i, V_j]\!] = 0$ for any $W \in \mathscr{W}[A_{ij}^\alpha]$, and hence $\exp(s\mathscr{L}_W)[\![V_i, V_j]\!] = 0$ for any $W \in \mathscr{W}[A_{ij}^\alpha]$. It thus follows that the commutation relations $[\![V_i, V_j]\!] = 0$ are stable under transport along all orbits of $\mathrm{ISO}[A_{ij}^\alpha]$.

Theorem 11.1. If $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$, then $\mathscr{H}^*[A^\alpha_{ij}]$ is an ideal of the Lie algebra $\text{ISO}[A^\alpha_{ij}]$; that is,

$$[\![\mathscr{H}^*[A^\alpha_{ij}], \mathscr{H}^*[A^\alpha_{ij}]]\!] \subset \mathscr{H}^*[A^\alpha_{ij}] \tag{11.12}$$

$$[\![\mathscr{H}^*[A^\alpha_{ij}], \text{ISO}[A^\alpha_{ij}]]\!] \subset \mathscr{H}^*[A^\alpha_{ij}] \tag{11.13}$$

Proof. The inclusion (11.12) follows directly from Lemma 11.2. Lemma 11.3 shows that

$$[\![s^i V_i, U]\!] = s^i [\![V_i, U]\!] - U\langle s^i\rangle V_i = \{s^i V_i\langle n^j\rangle - U\langle s^j\rangle\} V_j \tag{11.14}$$

for any $U \in \text{ISO}[A^\alpha_{ij}]$, from which I obtain the inclusion (11.13). ■

Theorem 11.2. If $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$, then the direct sum decomposition

$$\text{ISO}[A^\alpha_{ij}] = \mathscr{H}^*[A^\alpha_{ij}] \oplus \mathscr{W}[A^\alpha_{ij}] \tag{11.15}$$

induces the Lie algebra decomposition

$$[\![\mathscr{H}^*[A^\alpha_{ij}], \mathscr{H}^*[A^\alpha_{ij}]]\!] \subset \mathscr{H}^*[A^\alpha_{ij}] \tag{11.16}$$

$$[\![\mathscr{H}^*[A^\alpha_{ij}], \mathscr{W}[A^\alpha_{ij}]]\!] \subset \mathscr{H}^*[A^\alpha_{ij}] \tag{11.17}$$

$$[\![\mathscr{W}[A^\alpha_{ij}], \mathscr{W}[A^\alpha_{ij}]]\!] \subset \mathscr{W}[A^\alpha_{ij}] \tag{11.18}$$

Proof. The inclusions (11.16) and (11.17) follows directly from Theorem 11.1. Lemma 11.1 has established that $\text{ISO}[A^\alpha_{ij}]$ forms a Lie subalgebra of $T(K_1)$, and hence

$$[\![\mathscr{W}[A^\alpha_{ij}], \mathscr{W}[A^\alpha_{ij}]]\!] \subset \text{ISO}[A^\alpha_{ij}] \tag{11.19}$$

Since any element of $\mathscr{W}[A^\alpha_{ij}]$ is of the form

$$W_\eta = \eta^\alpha \partial_\alpha + V_i\langle \eta^\alpha\rangle \partial^i_\alpha \tag{11.20}$$

for $\{\eta^\alpha\}$ satisfying (10.3), it is clear that

$$[\![W_{\eta_1}, W_{\eta_2}]\!] \rfloor dx^i = 0 \tag{11.21}$$

Thus, since $V_j \rfloor dx^i = \delta^i_j$ and $\{V_i \mid 1 \le i \le n\}$ is a basis for $\mathscr{H}^*[A^\alpha_{ij}]$, the direct sum decomposition of $\text{ISO}[A^\alpha_{ij}]$ and (11.18) show that

$$[\![\mathscr{W}[A^\alpha_{ij}], \mathscr{W}[A^\alpha_{ij}]]\!] \cap \mathscr{H}^*[A^\alpha_{ij}] = \varnothing \tag{11.22}$$

This establishes the inclusion (11.17). ■

Remark. An explicit computation of the commutator of two elements

$$W_{\eta 1} = \eta^\alpha_1 \partial_\alpha + V_i\langle \eta^\alpha_1\rangle \partial^i_\alpha, \qquad W_{\eta_2} = \eta^\alpha_2 \partial_\alpha + V_i\langle \eta^\alpha_2\rangle \partial^i_2$$

of $\mathscr{W}[A^\alpha_{ij}]$ gives

$$[\![W_{\eta_1}, W_{\eta_2}]\!] = W_{\eta_3} \tag{11.23}$$

with

$$\eta_3^\alpha = W_{\eta_1}\langle \eta_2^\alpha \rangle - W_{\eta_2}\langle \eta_1^\alpha \rangle \tag{11.24}$$

This can be used to define a product on the collection of all solutions of (10.3) which converts this collection into a Poisson algebra.

12. TRANSPORT PROPERTIES

If U is any element of ISO$[A_{ij}^\alpha]$, the transport operator associated with U is denoted by

$$\mathcal{T}_U(s) = \exp(sU) \tag{12.1}$$

Lemma 12.1. If Ψ is a solution map of the horizontal ideal $\mathcal{H}[A_{ij}^\alpha] \in \mathfrak{H}(K_1)$ and $U \in$ ISO$[A_{ij}^\alpha]$, then $\mathcal{T}_U(s)\Psi$ is a solution map of the horizontal ideal $\mathcal{H}[A_{ij}^\alpha]$ for all s in a sufficiently small neighborhood of $s = 0$.

Proof. By definition, Ψ is a solution map of $\mathcal{H}[A_{ij}^\alpha]$ if and only if $\Psi^*\mu \neq 0$ and $\Psi^*\mathcal{H}[A_{ij}^\alpha] = 0$ (see Section 1). Now, $(\mathcal{T}_U(s)\Psi)^*\mu = \Psi^*\mathcal{T}_U^*(s)\mu \neq 0$ for s in a sufficiently small neighborhood of $s = 0$ by continuity of $\mathcal{T}(s)$ at $s = 0$. Similarly,

$$(\mathcal{T}(s)\Psi)^*\mathcal{H}[A_{ij}^\alpha] = \Psi^*\mathcal{T}_U^*(s)\mathcal{H}[A_{ij}^\alpha] = \Psi^*\mathcal{H}[A_{ij}^\alpha] = 0 \quad \blacksquare$$

Solution maps for systems of PDE are obtained as leaves of the foliation generated by an $\mathcal{H}[A_{ij}^\alpha] \in \mathfrak{H}(K_1)$, and leaf maps have the implicit presentation

$$g_\Sigma(x^i, q^\alpha, y_i^\alpha) = k_\Sigma, \qquad 1 \leq \Sigma \leq m_1 \tag{12.2}$$

where $\{g_\Sigma\}$ is a complete independent system of simultaneous first integrals of the system $V_i\langle g \rangle = 0$; that is, $\{g_\Sigma\}$ is any system of m_1 independent elements of $\mathcal{P}[A_{ij}^\alpha]$. The question thus arises as to the effects of the application of $\mathcal{T}_U(s)$ to elements of $\mathcal{P}[A_{ij}^\alpha]$.

Theorem 12.1. If $\mathcal{H}[A_{ij}^\alpha] \in \mathfrak{H}(K_1)$ and if f is an element of $\mathcal{P}[A_{ij}^\alpha]$, then

$$\mathcal{T}_V(s)\langle f \rangle = f \qquad \forall\, V \in \mathcal{H}^*[A_{ij}^\alpha] \tag{12.3}$$

and

$$\mathcal{T}_W(s)\langle f \rangle = g \in \mathcal{P}[A_{ij}^\alpha] \qquad \forall\, W \in \mathcal{W}[A_{ij}^\alpha] \tag{12.4}$$

Thus, $\mathcal{T}_U(s)$ is a map of $\mathcal{P}[A_{ij}^\alpha]$ into $\mathcal{P}[A_{ij}^\alpha]$ for all $U \in$ ISO$[A_{ij}^\alpha]$.

Proof. If $f \in \mathcal{P}[A_{ij}^\alpha]$, then $V_i\langle f \rangle = 0$, $1 \leq i \leq n$. These facts show that

$$\mathcal{T}_V(s)\langle f \rangle = \exp(s\mathcal{L}_V)\langle f \rangle = f$$

for all $V=n^iV_i\in\mathscr{H}^*[A^\alpha_{ij}]$. If $W\in\mathscr{W}[A^\alpha_{ij}]$, then (11.11) shows that $[\![V_i, W]\!]=0$, and hence $[\![V_i, \mathscr{T}_W(s)]\!]=0$. It therefore follows that

$$V_i\langle\mathscr{T}_W\langle f\rangle\rangle=[\![V_i, \mathscr{T}_W(s)]\!]\langle f\rangle+\mathscr{T}_W(s)\langle V_i\langle f\rangle\rangle=0 \tag{12.5}$$

Thus, if I define g by $g=\mathscr{T}_U(s)\langle f\rangle$, then (12.5) shows that $g\in\mathscr{P}[A^\alpha_{ij}]$. ■

The use of this theorem is as follows. A leaf of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$ can always be specified by equations of the form

$$g_\Sigma(x^i, q^\alpha, y^\alpha_i)=k_\Sigma, \qquad 1\le\Sigma\le m_1 \tag{12.6}$$

where $\{g_\Sigma\}$ is a system of m_1 independent elements of $\mathscr{P}[A^\alpha_{ij}]$. Thus, since $\mathscr{T}_U(s)$ acts on any element of $\mathscr{P}[A^\alpha_{ij}]$ as the identity operator for any $V\in\mathscr{H}^*[A^\alpha_{ij}]$, the relations (12.6) are taken into themselves by the action of $\mathscr{T}_V(s)$ for any $V\in\mathscr{H}^*[A^\alpha_{ij}]$. Thus, *transport by any element of $\mathscr{H}^*[A^\alpha_{ij}]$ takes any leaf of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$ into itself.* On the other hand, for $W\in\mathscr{W}[A^\alpha_{ij}]$,

$$\mathscr{T}_W(s)\langle g_\Sigma\rangle={}'g_\Sigma=F_W(g_\Gamma; s) \tag{12.7}$$

because ${}'g_\Sigma$ is an element of $\mathscr{P}[A^\alpha_{ij}]$ for every value of s, and any element of $\mathscr{P}[A^\alpha_{ij}]$ can be expressed as a function of the set $\{g_\Sigma\}$. Accordingly, the image of the leaf specification (12.6) under the action of $\mathscr{T}_W(s)$ for any $W\in\mathscr{W}[A^\alpha_{ij}]$ is given by

$$\mathscr{T}_W(s)\langle g_\Sigma\rangle={}'g_\Sigma=F_W(k_\Sigma; s) \tag{12.8}$$

This shows that *the action of any nontrivial element of $\mathscr{W}[A^\alpha_{ij}]$ will transport a leaf of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$ into another leaf of that foliation,* in general. Further, all x components of any $W\in\mathscr{W}[A^\alpha_{ij}]$ vanish, and hence $\mathscr{T}_W(s)$ leaves the base manifold M_n invariant. Thus, $\mathscr{T}_W(s)$ is a mapping of the fibers of K_1 over M_n (see Section 1). We can therefore paraphrase matters by saying that *transport by $\mathscr{H}^*[A^\alpha_{ij}]$ generates leaf automorphisms, while transport by $\mathscr{W}[A^\alpha_{ij}]$ generates fiber maps that interchange and reparametrize leaves.*

The balance ideal associated with a system of PDE on K_1 was shown in Section 6 to be of the form

$$\mathscr{B}_1[A^\alpha_{ij}]=I\{C^\alpha, H^\alpha_i, F_a\mu\} \tag{12.9}$$

with

$$F_a=h_a-V_i\langle W^i_a\rangle, \qquad 1\le a\le r \tag{12.10}$$

Since $\mathrm{ISO}[A^\alpha_{ij}]\circ\Psi$ is a solution map of $\mathscr{H}[A^\alpha_{ij}]$, (12.9) shows that any $U\in\mathrm{ISO}[A^\alpha_{ij}]$ will be an isovector of $\mathscr{B}_1[A^\alpha_{ij}]$ if and only if

$$\mathscr{L}_U F_a=U\langle F_a\rangle=L^b_aF_b \bmod \mathscr{H}[A^\alpha_{ij}], \qquad 1\le a\le r \tag{12.11}$$

However, any $U \in \mathrm{ISO}[A^\alpha_{ij}]$ is of the form

$$U = n^i V_i + \eta^\alpha \partial_\alpha + V_i\langle \eta^\alpha \rangle \partial^i_\alpha$$

and hence (12.11) are seen to be equations for the determination of the generating functions $\{n^i, \eta^\alpha\}$ such that $\{\eta^\alpha\}$ satisfies (9.3). On the other hand, I have shown that any element of $\mathscr{H}^*[A^\alpha_{ij}]$ maps any leaf of the foliation generated by $\mathscr{H}[A^\alpha_{ij}]$ into itself, and the graph of any solution of the given system of PDE is a leaf of the foliation generated by some admissible choice of $\{A^\alpha_{ij}\}$. I accordingly restrict attention to isovectors of the balance ideal that belong to $\mathscr{W}[A^\alpha_{ij}]$. If $W \in \mathscr{W}[A^\alpha_{ij}]$ and Ψ is a leaf map that solves the balance ideal (i.e., Ψ is the graph of a solution of the given system of PDE), then $\mathscr{T}_W(s) \circ \Psi$ is also a leaf map that solves the balance ideal for all values of s in a neighborhood of $s = 0$ when (12.11) hold. These results show that the study of isovectors of the horizontal have direct relevance in the search for solutions of PDE.

13. RESOLUTIONS

Let $\{g_\Sigma(x^i, q^\alpha, y^\alpha_i) \,|\, 1 \le \Sigma \le m_1\}$ be any system of m_1 independent elements of $\mathscr{P}[A^\alpha_{ij}]$. Such a system exists whenever $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$. It thus follows that

$$V_i\langle g_\Sigma \rangle = 0, \qquad \frac{\partial(g_\Sigma)}{\partial(q^\alpha, y^\alpha_i)} \neq 0 \tag{13.1}$$

because

$$V_i = \partial_i + y^\alpha_i \partial_\alpha + A^\alpha_{ij} \partial^j_\alpha \tag{13.2}$$

Accordingly, the coordinate transformation

$$R \,|\, x^{i'} = x^i, \qquad q^{\alpha'} = g^\alpha, \qquad y^{\alpha'}_i = g^\alpha_i \tag{13.3}$$

with

$$\{g^\alpha, g^\alpha_i \,|\, 1 \le \alpha \le N, 1 \le i \le n\} = \{g_\Sigma \,|\, 1 \le \Sigma \le m_1 = N(1+n)\}$$

belongs to $\mathrm{Diff}(K_1)$. The inverse of this coordinate transformation is easily seen to take the form

$$R^{-1} \,|\, x^i = x^{i'}, \qquad q^\alpha = Q^\alpha(x^{j'}, g_\Sigma), \qquad y^\alpha_i = Y^\alpha_i(x^{j'}, g_\Sigma) \tag{13.4}$$

The notation

$$\partial'_i = \partial/\partial x^{i'}, \qquad \partial'_\alpha = \partial/\partial q^{\alpha'}, \qquad \partial^{i'}_\alpha = \partial/\partial y^{\alpha'}_i \tag{13.5}$$

will be used for the natural basis for $T(K_1)$ relative to the prime coordinate system in what follows.

Theorem 13.1. If $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$, then the canonical basis for $\mathscr{H}^*[A^\alpha_{ij}]$ is resolved by the coordinate transformation R; that is

$$V'_i = R_* V_i = \partial'_i, \qquad 1 \le i \le n \tag{13.6}$$

Proof. The standard definition of R_* gives

$$R_* V_i = V_i\langle x^{j'}\rangle \partial'_j + V_i\langle q^{\alpha'}\rangle \partial'_\alpha + V_i\langle y^{\alpha'}_j\rangle \partial^{j'}_\alpha$$

the result then follows on noting that (13.3) imply

$$V_i\langle x^{j'}\rangle = \delta^j_i, \qquad V_i\langle q^{\alpha'}\rangle = V_i\langle y^{\alpha'}_j\rangle = 0 \quad \blacksquare$$

This theorem shows that the coordinate transformation R simultaneously "straightens out" all n elements of the canonical basis $\{V_i \mid 1 \le i \le n\}$ for $\mathscr{H}^*[A^\alpha_{ij}]$. On the other hand, the horizontal ideal $\mathscr{H}[A^\alpha_{ij}]$ is closed and is generated by the m_1 1-forms $\{C^\alpha, H^\alpha_i\}$. The Frobenius theorem thus shows that $\mathscr{H}[A^\alpha_{ij}]$ is also generated by m_1 exact 1-forms. As it turns out, the coordinate transformation R also provides a resolution of $\mathscr{H}[A^\alpha_{ij}]$ in terms of exact generating 1-forms.

Theorem 13.2. If $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$, then

$$R^* C^{\alpha'} = dg^\alpha = (\partial_\beta g^\alpha) C^\beta + (\partial^j_\beta g^\alpha) H^\beta_j \tag{13.7}$$

$$R^* H^{\alpha'}_i = dg^\alpha_i = (\partial_\beta g^\alpha_i) C^\beta + (\partial^j_\beta g^\alpha_i) H^\beta_j \tag{13.8}$$

where

$$C^{\alpha'} = dq^{\alpha'}, \qquad H^{\alpha'}_i = dy^{\alpha'}_i \tag{13.9}$$

Proof. Starting with the first of (13.9), the definition of R^* gives $R^* C^{\alpha'} = dg^\alpha$. Thus, use of (3.16) gives

$$R^* C^{\alpha'} = V_i\langle g^\alpha\rangle\, dx^i + (\partial_\beta g^\alpha) C^\beta + (\partial^j_\beta g^\alpha) H^\beta_j$$

However, $V_i\langle g^\alpha\rangle = 0$ and hence (13.7) is obtained. An exactly similar argument starting with the second of (13.9) gives (13.8). $\blacksquare$

The explicit resolution of the generators of $\mathscr{H}[A^\alpha_{ij}]$ can be obtained by noting that the system (13.7), (13.8) can be inverted because $R \in \mathrm{Diff}(K_1)$. Direct calculation will thus yield the relations

$$C^\alpha = N^\alpha_\beta\, dg^\beta + N^{\alpha j}_\beta\, dg^\beta_j, \qquad H^\alpha_i = N^\alpha_{i\beta}\, dg^\beta + N^{\alpha j}_{i\beta}\, dg^\beta_j \tag{13.10}$$

which are an explicit realization of the results of the Frobenius theorem.

These results provide a direct and possibly simpler realization of the basic existence theorem, Theorem 5.2. I have shown that the balance n-forms that characterize the system of PDE in K_1 have the presentation

$$B_a \equiv F_a \mu \bmod \mathscr{H}[A^\alpha_{ij}] \tag{13.11}$$

where

$$F_a(x^j, q^\beta, y_j^\beta) = h_a - V_i\langle W_a^i\rangle \tag{13.12}$$

Thus, if I define the functions $^*F_a$ by

$$^*F_a(x^{i'}, q^{\alpha'}, y_i^{\alpha'}) = R^{-1}\, ^*F_a \tag{13.13}$$

and use (13.3), then

$$^*F_a(x^{i'}, k^\alpha, k_i^\alpha) = 0, \qquad 1 \le a \le r \tag{13.14}$$

is the restriction of the equation $F_a = 0$ to the leaf of the foliation given by

$$g^\alpha(x^j, q^\beta, y_j^\beta) = k^\alpha, \qquad g_i^\alpha(x^j, q^\beta, y_j^\beta) = k_i^\alpha \tag{13.15}$$

Use of Theorem 5.2 thus yields the following result.

Theorem 13.3. If $\mathcal{H}[A_{ij}^\alpha] \in \mathfrak{H}(K_1)$ and if m_1 constants $\{k^\alpha, k_i^\alpha\}$ can be found such that the equations

$$^*F_a(x^{i'}, k_\alpha, k_i^\alpha) = 0, \qquad 1 \le a \le r \tag{13.16}$$

are satisfied on an open set $\mathcal{D} \subset M_n$, then the given system of PDE on K_1 has a solution whose graph is the set of points of K_1 in the leaf

$$g^\alpha(x^j, q^\beta, y_j^\beta) = k^\alpha, \qquad g_i^\alpha(x^j, q^\beta, y_j^\beta) = k_i^\alpha \tag{13.17}$$

over $\mathcal{D}$.

This theorem reduces the problem of solving a given system of PDE on K_1 to the problem of finding an open set $\mathcal{D}$ of roots of the system of r equations (13.16) in n variables $\{x^{i'}\}$. Accordingly, all of the integration procedures for the problem are contained in the construction of a system of m_1 independent integrals $\{g^\alpha, g_i^\alpha \mid 1 \le \alpha \le N, 1 \le i \le n\}$ of the system $\{V_i\langle g\rangle = 0\}$. When this theorem works, (13.17) show that *the solution is presented as a system of m_1 independent first integrals over $\mathcal{D}$.*

14. EXISTENCE AND PARAMETRIZATION OF SOLUTIONS TO THE SYSTEM (9.3)

The results obtained in the last section show that the horizontal ideal $\mathcal{H}[A_{ij}^\alpha] \in \mathfrak{H}(K_1)$ is generated by the m_1 exact 1-forms $\{dg^\alpha, dg_i^\alpha\}$. In order to use this fact to establish existence of solutions $\{\eta^\alpha\}$ of the system (9.3), I look at the problem of computing elements of $\mathrm{ISO}[A_{ij}^\alpha]$ in the new coordinate cover of K_1 that is generated by the coordinate transformation R of the previous section.

A vector field on K_1 in general position relative to the prime coordinate cover has the form

$$U' = n^{i'}\partial_i' + n^{\alpha'}\partial_\alpha' + n_i^{\alpha'}\partial_\alpha^{i'} \tag{14.1}$$

Theorem 14.1. A vector field U' is an isovector of the horizontal ideal $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K_1)$ in the prime coordinate system if and only if

$$n^{i'} = P^i(x^{j'}, q^{\beta'}, y_j^{\beta'}) \tag{14.2}$$

and

$$n^{\alpha'} = S^\alpha(q^{\beta'}, y_j^{\beta'}), \qquad n_i^{\alpha'} = T_i^\alpha(q^{\beta'}, y_j^{\beta'}) \tag{14.3}$$

for some smooth choice of the functions $\{P^i, S^\alpha, T_i^\alpha)$ of their indicated arguments.

Proof. Noting that $C^{\alpha'} = dq^{\alpha'}$, $H_i^{\alpha'} = dy_i^{\alpha'}$, an elementary calculation shows that

$$\mathscr{L}_{U'}C^{\alpha'} = dn^{\alpha'} = \partial_j' n^{\alpha'}\, dx^{j'} + \partial_\beta' n^{\alpha'} C^{\beta'} + \partial_\beta^{j'} n^{\alpha'} H_j^{\beta'} \tag{14.4}$$

and hence $\mathscr{L}_{U'}C^{\alpha'}$ will belong to $\mathscr{H}[A_{ij}^\alpha]$ only if $\partial_j' n^{\alpha'} = 0$. An exactly similar argument based on the relations

$$\mathscr{L}_{U'}H_i^{\alpha'} = dn_i^{\alpha'} \tag{14.5}$$

shows that $\mathscr{L}_{U'}H_i^{\alpha'}$ will belong to $\mathscr{H}[A_{ij}^\alpha]$ only if $\partial_{j'} n_i^{\alpha'} = 0$. Since there are no constraints imposed on the arguments of $n^{i'}$, I obtain (14.2) and (14.3). ■

Use of the prime coordinate system has significantly simplified the calculation of elements of $\mathrm{ISO}[A_{ij}^\alpha]$. In particular, there are no conditions such as (9.3) that have to be satisfied. Satisfaction of the conditions (9.3) in the original system of coordinates is thus implied by the forms that have been obtained for U' in $\mathscr{H}[A_{ij}^\alpha]$. This implies that the conditions (9.3) can always be satisfied in the original coordinate system; simply transform U' by the action of R_*^{-1} in order to obtain an isovector of $\mathscr{H}[A_{ij}^\alpha]$ in the original coordinate system. This isovector in the old coordinates will thus give a set of functions $\{\eta^\alpha \mid 1 \le \alpha \le N\}$ that will satisfy the system (9.3). The specifics of this calculation are as follows.

Any isovector of $\mathscr{H}[A_{ij}^\alpha]$ in the old coordinate system has the form

$$\begin{aligned} U &= n^i V_i + \eta^\alpha \partial_\alpha + V_i\langle \eta^\alpha \rangle \partial_\alpha^i \\ &= n^i \partial_i + (\eta^\alpha + n^j y_j^\alpha)\partial_\alpha + (V_i\langle \eta^\alpha\rangle + n^j A_{ji}^\alpha)\partial_\alpha^i \end{aligned} \tag{14.6}$$

and hence

$$U \rfloor dq^\alpha = \eta^\alpha + n^j y_j^\alpha \tag{14.7}$$

I now set

$$U = R_*^{-1} U' \tag{14.8}$$

to obtain

$$\eta^\alpha = (R_*^{-1} U') \rfloor dq^\alpha - n^j y_j^\alpha \tag{14.9}$$

An elementary calculation based on (13.4) gives

$$\begin{aligned}(R_*^{-1} U') \rfloor dq^\alpha &= n^{k'} \frac{\partial Q^\alpha}{\partial x^{k'}} + n^{\beta'} \frac{\partial Q^\alpha}{\partial q^{\beta'}} + n_j^{\beta'} \frac{\partial Q^\alpha}{\partial y_j^{\beta'}} \\ &= n^{k'} \frac{\partial Q^\alpha}{\partial x^{k'}} + n^{\beta'} \frac{\partial Q^\alpha}{\partial g^\beta} + n_j^{\beta'} \frac{\partial Q^\alpha}{\partial g_j^\beta}\end{aligned} \tag{14.10}$$

and

$$n^i = n^{i'} \tag{14.11}$$

Now, (14.2) and (14.3) give the evaluations

$$\begin{gathered} n^{i'} = P^i(x^j, g^\beta, g_j^\beta) \\ n^{\alpha'} = S^\alpha(g^\beta, g_j^\beta), \qquad n_i^{\alpha'} = T_i^\alpha(g^\beta, g_j^\beta) \end{gathered} \tag{14.12}$$

and hence explicit evaluation of the set $\{\eta^\alpha \mid 1 \le \alpha \le N\}$ can be obtained by combining (14.9)-(14.12). The functions $\{Q^\alpha(x^j, g^\beta, g_j^\beta) \mid 1 \le \alpha \le N\}$ in these evaluations are fixed by the transformation R. On the other hand, the functions $\{P^i, S^\alpha, T_i^\alpha \mid 1 \le i \le n, 1 \le \alpha \le N\}$ are arbitrary functions of their indicated arguments. These functions therefore parametrize the possible choices of the functions $\{\eta^\alpha\}$, and hence they parametrize the solutions of (9.3).

An identical calculation can be used to compute independently the y_i^α components of an isovector of $\mathscr{H}[A_{ij}^\alpha]$. Although they are of an equally complicated nature as those obtained above for the q^α components, they turn out to be exactly $V_i\langle\eta^\alpha\rangle$ when the known (Edelen, 1980) transformation properties of the y's are used. The results that obtain from the calculation given above are thus self-consistent and complete.

Partial results of this nature could be anticipated from the form of the system (9.3),

$$V_i V_j \langle \eta^\alpha \rangle = (\partial_\beta A_{ij}^\alpha) \eta^\beta + (\partial_\beta^k A_{ij}^\alpha) V_k \langle \eta^\beta \rangle$$

Simply observe that if $\partial_\beta A_{ij}^\alpha$ all vanish, then any set of N elements of $\mathscr{P}[A_{ij}^\alpha]$ will work; that is, $\eta^\alpha = N^\alpha(g_\Sigma) = N^\alpha(g^\beta, g_j^\beta)$.

15. ISOVECTORS OF HORIZONTAL IDEALS FOR SECOND-ORDER CONTACT MANIFOLDS

Previous sections of this paper have dealt exclusively with first-order contact manifolds K_1. Similar results hold for contact manifolds of any finite order. An indication of the structure of these results is provided by the study of contact manifolds of second order. I will simply state the results, since the proofs follow exactly the same lines as those for corresponding results on K_1.

Recall that the horizontal ideal on K_2 is generated by the 1-forms

$$C^\alpha = dq^\alpha - y^\alpha_k\, dx^k, \qquad C^\alpha_i = dy^\alpha_i - y^\alpha_{ik}\, dx^k \tag{15.1}$$

$$H^\alpha_{ij} = dy^\alpha_{ij} - B^\alpha_{ijk}\, dx^k \tag{15.2}$$

in which case I write $\mathscr{H}[B^\alpha_{ijk}]$ for the horizontal ideal associated with $\{B^\alpha_{ijk}\}$.

Theorem 15.1. Any isovector of $\mathscr{H}[B^\alpha_{ijk}] \in \mathfrak{H}(K_2)$ is of the form

$$U = n^i V_i + \eta^\alpha \partial_\alpha + V_i\langle \eta^\alpha\rangle \partial^i_\alpha + V_i V_j \langle \eta^\alpha \rangle \partial^{ij}_\alpha \tag{15.3}$$

for any choice of $\{\eta^\alpha \in \Lambda^0(K_2) \,|\, 1 \le \alpha \le N\}$ such that

$$\begin{aligned} V_k V_i V_j \langle \eta^\alpha \rangle = (\partial_\beta B^\alpha_{ijk})\eta^\beta + (\partial^m_\beta B^\alpha_{ijk}) V_m \langle \eta^\beta \rangle \\ + (\partial^{mr}_\beta B^\alpha_{ijk}) V_m V_r \langle \eta^\beta \rangle \end{aligned} \tag{15.4}$$

Theorem 15.2. The collection $\mathrm{ISO}[B^\alpha_{ijk}]$ of all isovectors of $\mathscr{H}[B^\alpha_{ijk}]$ admits the direct sum decomposition

$$\mathrm{ISO}[B^\alpha_{ijk}] = \mathscr{H}^*[B^\alpha_{ijk}] \oplus \mathscr{W}[B^\alpha_{ijk}] \tag{15.5}$$

where $\mathscr{W}[B^\alpha_{ijk}]$ is the collection of all vectors of the form

$$W_\eta = \eta^\alpha \partial_\alpha + V_i \langle \eta^\alpha \rangle \partial^i_\alpha + V_i V_j \langle \eta^\alpha \rangle \partial^{ij}_\alpha \tag{15.6}$$

for any $\{\eta^\alpha \in \Lambda^0(K_2) \,|\, \le \alpha \le N\}$ that satisfies (15.4).

Theorem 15.3. The collection $\mathrm{ISO}[B^\alpha_{ijk}]$ forms a Lie algebra that admits the subalgebra $\mathscr{H}^*[B^\alpha_{ijk}]$ as an ideal:

$$[\![\mathscr{H}^*[B^\alpha_{ijk}], \mathscr{H}^*[B^\alpha_{ijk}]]\!] \subset \mathscr{H}^*[B^\alpha_{ijk}] \tag{15.7}$$

$$[\![\mathscr{H}^*[B^\alpha_{ijk}], \mathscr{W}[B^\alpha_{ijk}]]\!] \subset \mathscr{H}^*[B^\alpha_{ijk}] \tag{15.8}$$

$$[\![\mathscr{W}[B^\alpha_{ijk}], \mathscr{W}[B^\alpha_{ijk}]]\!] \subset \mathscr{W}[B^\alpha_{ijk}] \tag{15.9}$$

16. EXTENDED CANONICAL TRANSFORMATIONS FOR FIRST-ORDER CONTACT MANIFOLDS

The coordinate transformations considered in Section 13 do not preserve the structure of the contact 1-forms (i.e., $C^\alpha = dq^\alpha - y^\alpha_k\, dx^k$, while

$C^{\alpha'} = dq^{\alpha'}$). The analogy with classical contact transformations that obtains for $n = 1$ or $N = 1$ suggests that an analysis of transformations that preserves the structure of the generating 1-forms of a horizontal ideal will prove to be useful.

The contact manifold K_1 has a system of local coordinates $\{x^i, q^\alpha, y_i^\alpha\}$ and a system of m_1 1-forms

$$C^\alpha = dq^\alpha - y_k^\alpha\, dx^k, \qquad H_i^\alpha = dy_i^\alpha - A_{ik}^\alpha\, dx^k \tag{16.1}$$

with

$$A_{ik}^\alpha = A_{ki}^\alpha \tag{16.2}$$

These serve to define the horizontal ideal

$$\mathscr{H}[A_{ij}^\alpha] = I\{C^\alpha, H_i^\alpha \mid 1 \le \alpha \le N, 1 \le i \le n\} \tag{16.3}$$

of $\Lambda(K_1)$. Let $'K_1$ be a replica of K_1 with a system of local coordinates $\{'x^i, 'q^\alpha, 'y_i^\alpha\}$ and a system of m_1 1-forms

$$'C^\alpha = d('q^\alpha) - 'y_k^\alpha\, d('x^k), \qquad 'H_i^\alpha = d('y_i^\alpha) - 'A_{ik}^\alpha\, d('x^k) \tag{16.4}$$

with

$$'A_{ik}^\alpha = 'A_{ki}^\alpha \tag{16.5}$$

These serve to define the horizontal ideal

$$'\mathscr{H}['A_{ij}^\alpha] = \{'C^\alpha, 'H_i^\alpha \mid 1 \le \alpha \le N, 1 \le i \le n\} \tag{16.6}$$

of $\Lambda('K_1)$.

Definition 16.1. A map $S: K_1 \to 'K_1$ that belongs to $\mathrm{Diff}(K_1, 'K_1)$ is an *extended canonical transformation* if and only if

$$S^*\, '\mathscr{H}['A_{ij}^\alpha] \subset \mathscr{H}[A_{ij}^\alpha] \tag{16.7}$$

The collection of all extended canonical transformations is denoted by

$$\mathrm{Ect} = \{S \in \mathrm{Diff}(K_1, 'K_1) \mid S^*\, '\mathscr{H}['A_{ij}^\alpha] \subset \mathscr{H}[A_{ij}^\alpha]\} \tag{16.8}$$

Theorem 16.1. Let $\mathscr{H}[A_{ij}^\alpha]$ be a horizontal ideal of $\Lambda(K_1)$ and let $\{V_i \mid 1 \le i \le n\}$ be the canonical basis for $\mathscr{H}^*[A_{ij}^\alpha]$. A transformation $S \in \mathrm{Diff}(K_1, 'K_1)$, with the presentation

$$'x^i = s^i(x^j, q^\beta, y_j^\beta), \qquad 'q^\alpha = s^\alpha(x^j, q^\beta, y_j^\beta) \tag{16.9}$$

$$'y_i^\alpha = s_i^\alpha(x^j, q^\beta, y_j^\beta) \tag{16.10}$$

is an extended canonical transformation if and only if

$$\det(V_i\langle s^j\rangle) \ne 0 \tag{16.11}$$

$$V_j\langle s^\alpha\rangle = s_k^\alpha V_j\langle s^k\rangle \tag{16.12}$$

$$^*A_{km}^\alpha V_j\langle s^m\rangle V_i\langle s^k\rangle = V_j V_i\langle s^\alpha\rangle - s_k^\alpha V_j V_i\langle s^k\rangle \tag{16.13}$$

$$[\![V_i, V_j]\!]\langle s^\alpha\rangle = [\![V_i, V_j]\!]\langle s^k\rangle = 0 \tag{16.14}$$

where

$$^*A_{ij}^\alpha = S^*\,'A_{ij}^\alpha \tag{16.15}$$

When these conditions are satisfied, we have

$$S^*\,'C^\alpha = (\partial_\beta s^\alpha - s_k^\alpha \partial_\beta s^k)C^\beta + (\partial_\beta^j s^\alpha - s_k^\alpha \partial_\beta^j s^k)H_j^\beta \tag{16.16}$$

$$S^*\,'H_i^\alpha = (\partial_\beta s_i^\alpha - {}^*A_{ik}^\alpha \partial_\beta s_i^\alpha)C^\beta + (\partial_\beta^j s_i^\alpha - {}^*A_{ik}^\alpha \partial_\beta^j s_i^\alpha)H_j^\beta \tag{16.17}$$

Proof. An elementary calculation shows that (16.9) and (16.10) yield

$$S^*\,'C^\alpha = ds^\alpha - s_k^\alpha\, ds^k \tag{16.18}$$

when (3.16) is used to evaluate the indicated exterior derivatives, I obtain

$$S^*\,'C^\alpha = (V_j\langle s^\alpha\rangle - s_k^\alpha V_j\langle s^k\rangle)\, dx^j + (\partial_\beta s^\alpha - s_k^\alpha \partial_\beta s^k)C^\beta$$
$$+ (\partial_\beta^j s^\alpha - s_k^\alpha \partial_\beta^j s^k)H_j^\beta \tag{16.19}$$

Thus, S can be an extended canonical transformation only when (6.12) holds, in which case I obtain (6.16). A similar calculation based on

$$S^*\,'H_i^\alpha = ds_i^\alpha - {}^*A_{ik}^\alpha\, ds^k \tag{16.20}$$

shows that

$$V_j\langle s_i^\alpha\rangle = {}^*A_{ik}^\alpha V_j\langle s^k\rangle \tag{16.21}$$

must be satisfied, in which case I obtain (16.17). When (16.12) is used to evaluate $V_j V_i\langle s^\alpha\rangle$ and (16.21) is used to simplify the results, I obtain (16.13). Conversely, it is easily seen that (16.12) and (16.13) imply (16.21). Now, (6.12) serves to determine the functions s_i^α, while (6.13) serves to determine the functions $^*A_{ij}^\alpha$ provided (6.11) holds. I will then have the required symmetry $'A_{ij}^\alpha = {}'A_{ji}^\alpha$ only if $^*A_{ij}^\alpha = {}^*A_{ji}^\alpha$, and (16.13) shows that this will be the case only when the functions s^α and s^i satisfy the conditions (16.14). ■

Satisfaction of the condition (16.11) implies the existence of functions S_j^i such that

$$S_j^i V_i\langle s^k\rangle = S_i^k V_j\langle s^i\rangle = \delta_j^k \tag{16.22}$$

Accordingly, (16.12) gives the explicit evaluations

$$s_j^\alpha = S_j^m V_m\langle s^\alpha\rangle \tag{16.23}$$

while (16.13) and (16.23) yield

$$^*A_{ij}^\alpha = S_j^m S_i^r \{ V_m V_r\langle s^\alpha\rangle - S_t^k V_k\langle s^\alpha\rangle V_m V_r\langle s^t\rangle\} \tag{16.24}$$

Since $S \in \text{Diff}(K_1, 'K_1)$, the inverse mapping S^{-1} exists. I can therefore use $*A^\alpha_{ij} = S^* \,'A^\alpha_{ij}$ to obtain the explicit evaluations

$$'A^\alpha_{ij} = S^{-1*}(S^m_j S^r_i \{V_m V_r\langle s^\alpha\rangle - S^k_t V_k\langle s^\alpha\rangle V_m V_r\langle s^t\rangle\}) \tag{16.25}$$

These explicit evaluations show that *any extended canonical transformation is determined by specification of the $n + N$ functions $\{s^\alpha, s^i \mid 1 \le \alpha \le N, 1 \le i \le n\}$ of $\{x^j, q^\beta, y^\beta_j\}$ that satisfy the conditions* (16.11), (16.14) *and are such that* $S \in \text{Diff}(K_1, 'K_1)$.

Definition 16.2. Any system of functions $\{s^\alpha, s^i\}$ of the arguments $\{x^j, q^\beta, y^\beta_j\}$ that satisfy the conditions (16.11) and (16.14), and are such that $S \in \text{Diff}(K_1, 'K_1)$, will be referred to as *generating functions* of an extended canonical transformation.

The analysis presented so far has dealt with extended canonical transformations of a general horizontal ideal of K_1. We now specialize to closed horizontal ideals.

Theorem 16.2. If $\mathscr{H}[A^\alpha_{ij}]$ is a closed ideal of $\Lambda(K_1)$ (i.e., $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$), then any extended canonical transformation is generated by generating functions $\{s^\alpha, s^i\}$ such that (16.11) is satisfied, $S \in \text{Diff}(K_1, 'K_1)$, and the matrix of coefficient functions that multiply $\{C^\beta, H^\beta_j\}$ in (16.16) and (16.17) is a nonsingular matrix. Under these conditions,

$$S^* \,'\mathscr{H}['A^\alpha_{ij}] = \mathscr{H}[A^\alpha_{ij}] \tag{16.26}$$

and $'\mathscr{H}['A^\alpha_{ij}]$ is a closed ideal of $\Lambda('K_1)$; that is, $'\mathscr{H}['A^\alpha_{ij}] \in \mathfrak{H}('K_1)$.

Proof. If $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$, then Theorems 2.1 and 3.2 show that $[\![V_i, V_j]\!] = 0$. The conditions (6.14) are therefore satisfied for any choice of the generating functions $\{s^\alpha, s^i\}$. Thus, the generating functions only have to satisfy (16.11) and be such that $S \in \text{Diff}(K_1, 'K_1)$. I now use (16.16) and (16.17) and $\mathscr{H}[A^\alpha_{ij}] \in \mathfrak{H}(K_1)$ to obtain

$$S^* d('C^\alpha) = dS^* \,'C^\alpha \equiv 0 \bmod \mathscr{H}[A^\alpha_{ij}] \tag{16.27}$$

$$D^* d('H^\alpha_i) = dS^* \,'H^\alpha_i \equiv 0 \bmod \mathscr{H}[A^\alpha_{ij}] \tag{16.28}$$

By definition, any extended canonical transformation S is such that $S^* \,'\mathscr{H}['A^\alpha_{ij}] \subset \mathscr{H}[A^\alpha_{ij}]$. Satisfaction of the additional hypothesis concerning the nonsingularity of the matrix of coefficients of the right-hand sides of equations (16.16) and (16.17) thus implies $S^* \,'\mathscr{H}['A^\alpha_{ij}] = \mathscr{H}[A^\alpha_{ij}]$. The inverse transformation S^{-1} exists because $S \in \text{Diff}(K_1, 'K_1)$, and hence $S^{-1*}\mathscr{H}[A^\alpha_{ij}] = '\mathscr{H}['A^\alpha_{ij}]$ under the given hypotheses. Application of S^{-1*} to

(16.27) and (16.28) serves to establish

$$d('C^\alpha) \equiv 0, \qquad d('H_i^\alpha) \equiv 0 \bmod '\mathscr{H}['A_{ij}^\alpha]$$

that is, $'\mathscr{H}['A_{ij}^\alpha] \in \mathfrak{H}('K_1)$. ■

This theorem shows how to construct many closed horizontal ideals of $\Lambda('K_1)$. Simply take any collection of functions $\{s^\alpha, s^i\} \subset \Lambda^0(K_1)$ that satisfies the conditions of Theorem 16.2 and any $\{A_{ij}^\alpha\}$ such that $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K_1)$. Evaluations of $'A_{ij}^\alpha$ such that $'\mathscr{H}['A_{ij}^\alpha] \in \mathfrak{H}('K_1)$ will then be given by (16.25). This construction partially answers the question of how to construct closed horizontal ideals that underlies the existence proofs given in Sections 5 and 6. Indeed, the subset of $\mathfrak{H}('K_1)$ that can be generated in this manner is extensive, in view of the fact that we know that all assignments $A_{ij}^\alpha = \partial_i \partial_j \xi^\alpha(x^k)$ will generate elements of $\mathfrak{H}(K_1)$.

Let $'\mathscr{H}^*['A_{ij}^\alpha]$ be the module of Cauchy characteristic vector fields of $'\mathscr{H}['A_{ij}^\alpha]$, and let $\{'V_i \mid 1 \le i \le n\}$ be the canonical basis; that is,

$$'V_i = '\partial_i + 'y_i^\alpha\, '\partial_\alpha + 'A_{ij}^\alpha\, '\partial_\alpha^j \tag{16.29}$$

On the other hand

$$S_* V_i = V_i\langle s^k\rangle\, '\partial_k + V_i\langle s^\beta\rangle\, '\partial_\beta + V_i\langle s_j^\beta\rangle\, '\partial_\beta^j \tag{16.30}$$

When (16.12) and (16.21) are used, (16.30) can be written in the equivalent form

$$S_* V_i = V_i\langle s^k\rangle\{'\partial_k + s_k^\beta\, '\partial_\beta + {}^*A_{kj}^\beta\, '\partial_\beta^j\} \tag{16.31}$$

for any $S \in \text{Ect}$. Thus, since $'y_k^\beta = s_k^\beta$, $\det(V_i\langle s^j\rangle) \neq 0$ for any extended canonical transformation, and the right-hand sides of (16.31) have to be composed with S^{-1}, we have established the following result.

Theorem 16.3. If S is an extended canonical transformation, then the canonical bases for $\mathscr{H}^*[A_{ij}^\alpha]$ and $'\mathscr{H}^*['A_{ij}^\alpha]$ are related by

$$S_* V_i = (V_i\langle s^k\rangle \circ S^{-1}\, 'V_k \tag{16.32}$$

and hence

$$S_* \mathscr{H}^*[A_{ij}^\alpha] = '\mathscr{H}^*['A_{ij}^\alpha] \tag{16.33}$$

It is of interest to note in passing that $S_* V_i = 'V_i$ only when $V_i\langle s^j\rangle = \delta_i^j$.

These results give procedures for constructing systems of partial differential equations for which large families of solutions can be constructed. For simplicity, I will take $n = 2$ and use $\{'x, 't\}$ as a system of local coordinates on the base manifold $'M_2$ of $'K_1$. Let $\{'h^\alpha \mid 1 \le \alpha \le N\}$ be a system of N elements of $\Lambda^0('K_1)$. In order to make matters specific, I consider the balance 2-forms

$$'B^\alpha = 'h^\alpha d('x) \wedge d('t) - d('y_t^\alpha) \wedge d('t), \qquad 1 \le \alpha \le N \tag{16.34}$$

that characterize the system of PDE (coupled wave equations in characteristic coordinates)

$$\frac{\partial^2 \phi^\alpha}{\partial('x)\partial('t)} = h^\alpha\left('x, 't, \phi^\beta, \frac{\partial \phi^\beta}{\partial('x)}, \frac{\partial \phi^\beta}{\partial('t)}\right)$$

Previously established results show that

$$'B^\alpha \equiv 'F^\alpha \, d('x) \wedge d('t) \bmod '\mathscr{H}['A_{ij}^\alpha] \tag{16.35}$$

with

$$'F^\alpha = 'h^\alpha - 'V_x\langle 'y_t^\alpha \rangle = 'h^\alpha - 'A_{xt}^\alpha \tag{16.36}$$

Theorem 6.4 shows that $'\mathscr{H}['A_{ij}^\alpha] \in \mathfrak{H}_s('K_1)$ if the $'A$'s are generated in the manner specified above from some $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K_1)$ and

$$'h^\alpha = 'A_{xt}^\alpha \tag{16.37}$$

In this event, every leaf of the foliation generated by $'\mathscr{H}['A_{ij}^\alpha]$ is the graph of a solution map of the given system of PDE. Theorems 6.4 and 6.5 can also be used to construct more general systems of PDE for which the existence of solution maps can be established.

This procedure can be turned around to give an associated inverse problem. In order to do this, I make any appropriate choice of the generating functions $\{s^\alpha, s^i\}$ that satisfy the hypotheses of Theorem 16.2, choose any $\mathscr{H}[A_{ij}^\alpha] \in \mathfrak{H}(K_1)$, and then use (16.25) to compute the functions $'A_{ij}^\alpha$. Suppose that this calculation gives $'A_{xt}^\alpha = G^\alpha('x, 't, 'q^\beta, 'y_x^\beta, 'y_t^\beta)$. The system of PDE (16.35) for which

$$'h^\alpha = G^\alpha('x, 't, 'q^\beta, 'y_x^\beta, 'y_t^\beta) \tag{16.38}$$

will then be such that every leaf of the foliation generated by $'\mathscr{H}['A_{ij}^\alpha]$ will be the graph of a solution map. It is interesting to note that equation (16.37), when $'A_{xt}^\alpha$ is evaluated in terms of the generating functions $\{s^\alpha, s^i\}$ by use of (16.25), can be viewed as a system of partial differential equations for the determination of the functions $\{s^\alpha, s^i\}$ for given $'h^\alpha \in \Lambda^0('K_1)$. If these equations have solutions, then the existence of solutions to the given system of PDE is assured, and the graphs of the solutions are all leaves of the foliation generated by $'\mathscr{H}['A_{ij}^\alpha]$. Concatenations of these ideas with integration of the orbital equations associated with the canonical basis $\{'V_i\}$ gives a structure that is similar in some respects to classical Hamilton–Jacobi theory.

Corresponding results for second-order contact manifolds can be obtained by use of the same arguments as those just given for K_1.

APPENDIX. SEQUENTIAL INTEGRATION OF THE ORBITAL EQUATIONS OF VECTOR FIELDS IN JACOBI NORMAL FORM AND REPRESENTATIONS OF THE ASSOCIATED FOLIATION STRUCTURE

For the purposes of this discussion, let K be an M-dimensional space with local coordinates $\{z^A \mid 1 \leq A \leq M\}$ and let

$$V_i = V_i^A(z^B)\frac{\partial}{\partial z^A}, \qquad 1 \leq i \leq n < M \tag{A1}$$

be a system of smooth vector fields on K in Jacobi normal form; that is,

$$[V_i, V_j] = 0 \tag{A2}$$

If $P_0: \{z_0^A\}$ is a point in K, then the fundamental existence and uniqueness theorem for systems of autonomous ordinary differential equations shows that there exists a neighborhood $\mathbb{J}_1$ of $u^1 = 0$ in $\mathbb{R}$ on which the initial value problem

$$\frac{dZ_1^A}{du^1} = V_1^A(Z_1^B), \qquad Z_1^A(0) = z_0^A \tag{A3}$$

is satisfied. Let the solution of this initial value problem be denoted by

$$Z_1^A = \mathscr{X}_1^A(P_0; u^1) \tag{A4}$$

The functions $\{\mathscr{X}_1^A\}$ serve to define a map $\Psi_1: \mathbb{J}_1 \to K$ by

$$\Psi_1 \mid z^A = \mathscr{X}_1^A(P_0; u^1) \tag{A5}$$

The same existence theorem shows that the initial value problems

$$\frac{dZ_2^A}{du^2} = V_2^A(Z_2^B), \qquad Z_2^A(0) = \mathscr{X}_2^A(P_0; u^1) \tag{A6}$$

are solvable on an open set $\mathbb{J}_2$ of $\mathbb{R}^2$ that contains the point $u^1 = u^2 = 0$. Let the solutions of this problem be denoted by

$$Z_2^A = \mathscr{X}_2^A(P_0; u^1, u^2) \tag{A7}$$

We then have a map $\Psi_2: \mathbb{J}_2 \to K$ that is defined by

$$\Psi_2 \mid z^A = \mathscr{X}_2^A(P_0; u^1, u^2) \tag{A8}$$

Since $\{V_i\}$ are in Jacobi normal form, all of the flows generated by the V's commute with each other. Accordingly, the representation (A8) will be the same if we were to start with V_2 and then use V_1.

Continuing in this manner, we obtain a map $\Psi = \Psi_n : \mathbb{J}_n \subset \mathbb{R}^n \to K$ with the representation

$$\Psi \mid z^A = \mathscr{Z}_n^A(P_0; u^1, u^2, \ldots, u^n) \tag{A9}$$

This map is said to be obtained by *sequential integration* of the orbital equations of the system $\{V_i\}$ starting from the point P_0. By construction, the point P_0 is in the range of Ψ. In fact, P_0 is the image of the origin of $\mathbb{R}^n$. Since the vectors $\{V_i\}$ are in Jacobi normal form, the representation (A9) is independent of the order in which we select the V's in the sequential integration process.

Since the system $\{V_i\}$ is in Jacobi normal form, the system of simultaneous, first-order PDE

$$V_i\langle g\rangle = 0, \qquad 1 \le i \le n \tag{A10}$$

will have solutions that can be expressed as functions of $M - n$ independent primitive integrals $\{g_\Sigma(z^A) \mid 1 \le \Sigma \le M - n\}$. The space K is thus foliated by manifolds of dimension n that are implicitly defined by

$$g_\Sigma(z^A) = k_\Sigma, \qquad 1 \le \Sigma \le M - n \tag{A11}$$

The leaf $\mathscr{L}(P_0)$ of this foliation that passes through the point P_0 is given by choosing the constants $\{k_\Sigma\}$ by

$$k_\Sigma = g_\Sigma(z_0^A) \tag{A12}$$

The map Ψ that is constructed by sequential integration of $\{V_i\}$ starting from P_0 is then easily seen to be a map from $\mathbb{J}_n \subset \mathbb{R}^n$ into $\mathscr{L}(P_0)$. In fact, Ψ gives a local parametric representation of $\mathscr{L}(P_0)$.

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